where, throughout this note E denotes "the expected value of" and var. "the variance of": let us also assume that the probability distribution of ε_i is normal.

Let X_i be the general term of the disturbance series, generated according to the law

$$\sum_{s=0}^{m} b_s (X_{i-s} - c) = \varepsilon_i \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.2)$$

where $b_0 = 1$ and where i ranges from minus infinity to plus infinity.

Given the sequence $X_1 \ldots X_n$, $(n > 2m)$, and knowing $b_1 \ldots b_m$, we require to estimate the central value c. Any arithmetic mean, weighted or not, of $X_1 \ldots X_n$ will give an unbiased estimator of c, but the unweighted arithmetic mean will not in general be most efficient. To find the most efficient estimator we may use the method of maximum likelihood.

We may regard $X_1 \ldots X_n$ as linear functions of n independent variables $\eta_1 \ldots \eta_n$ such that

$$E(\eta_i) = 0, \quad \text{var.}\,(\eta_i) = \alpha^2 \qquad i = 1, 2 \ldots n \quad . \quad . \quad . \quad . \qquad (3.3)$$

where the probability distribution of each η_i is normal. One such set η_i would be of the form

$$\begin{aligned} \eta_1 &= k_{11}(X_1 - c) \\ \eta_2 &= k_{21}(X_1 - c) + k_{22}(X_2 - c) \\ &\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad . \quad . \qquad (3.4) \\ \eta_m &= k_{m1}(X_1 - c) + k_{m2}(X_2 - c) + \ldots k_{mm}(X_m - c) \\ \eta_i &= \varepsilon_i \qquad \text{for } m < i \leqslant n \end{aligned}$$

the k_{ij} being so chosen as to ensure the independence of the η_i and the satisfaction of (3.3).

It will then be seen that

$$\sum_{i=1}^{n} \eta_i^2 = \sum_{r=0}^{m} \sum_{s=0}^{m} b_r b_s \sum_{t=r+s+1}^{n} (X_{t-r} - c)(X_{t-s} - c) \quad . \quad . \quad . \qquad (3.5)$$

The reader need only satisfy himself that this formula gives the correct coefficients for each product $(X_u - c)(X_v - c)$ wherein $u > m$, for since the formula remains unchanged if for every X_u we substitute X_{n+1-u} it must then give the correct coefficients for every product $(X_u - c)(X_v - c)$ wherein $u \leqslant n - m$.

It will be convenient to denote any expression of the form

$$\sum_{r=0}^{m} \sum_{s=0}^{m} b_r b_s \sum_{t=r+s+1}^{n} (X_{t-r}\, Y_{t-s}) \text{ by } B(X, Y) \quad . \quad . \quad . \quad . \quad . \qquad (3.6)$$

In this notation we may write (3.5) as

$$\sum_{i=1}^{n} \eta_i^2 = B(X - c, X - c) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.7)$$

Now let L be the likelihood of our sequence $X_1 \ldots X_n$. Then

$$\log L = -\frac{n}{2}\log 2\pi - n\log\alpha - \frac{1}{2\alpha^2}B(X - c, X - c) + \log J \quad . \qquad (3.8)$$

where J is the jacobian

$$\frac{\partial(\eta_2, \ldots, \eta_n)}{\partial(X_1, \ldots, X_n)}$$

a function of $b_0\, b_1 \ldots b_m$ only.

In order to find c^* the maximum likelihood estimator for c, we equate to zero the derivative *w.r.t.c* of our expression for $\log L$. We obtain

$$B(X - c^*, 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.9)$$