Advances in Box-Jenkins Modeling
2. Applications

ANGUS IAN MCLEOD

Department of Statistics, University of Waterloo, Waterloo, Ontario, Canada

KEITH WILLIAM HIPEL

Department of Systems Design, University of Waterloo, Waterloo, Ontario, Canada

WILLIAM C. LENNOX

Department of Civil Engineering, University of Waterloo, Waterloo, Ontario, Canada

Recent Box-Jenkins techniques are employed to determine both nonseasonal and seasonal models for actual time series. The applied examples are carefully explained in order to demonstrate the utility of the new procedures that have been developed for use at the identification, estimation, and diagnostic check stages of model development. Even though more methods are now available for model building, it is demonstrated that this fact enhances rather than complicates the model construction phases. Furthermore, for all three applications considered, better models are obtained than it was previously possible to obtain. A new technique is described for optimal forecasting of the original time series when the data have been transformed by a nonlinear transformation.

INTRODUCTION

Improved Box-Jenkins modeling procedures are available to simplify model construction. As is explained by *Hipel et al.* [1977*a*], new methods have been developed for use at the identification, estimation, and diagnostic check stages of model development. The purpose of this paper is to demonstrate the relative ease with which improved models can be obtained in practical applications by employing recent modeling procedures.

A constrained nonseasonal autoregressive integrated moving average (Arima) model is determined for average annual river flows, a nonmultiplicative Arima process is fit to a yearly sunspot number series, and a multiplicative seasonal Box-Jenkins model is found for modeling monthly international airline passenger data. In all three cases, better models are obtained than those previously cited in the literature. It is shown at the identification stage how a simultaneous inspection of the autocorrelation function (ACF), partial autocorrelation function (PACF), inverse autocorrelation function (IACF), and inverse partial autocorrelation function (IPACF) leads to a quick but usually accurate initial design for a tentative model to estimate. All four identification procedures are used for identifying both the nonseasonal and the seasonal model. At the estimation stage, more efficient parameter estimates are procured by using the modified sum of squares method [*McLeod,* 1976*a*]. In particular, better parameter estimates than those calculated by using the unconditional sum of squares technique [*Box and Jenkins,* 1970, chapter 7] are obtained for the moving average (MA) terms of the seasonal Arima model that is fit to the airline passenger data. The Akaike information criterion (AIC) is used for model discrimination purposes [*Akaike,* 1974] for both the yearly river flow and the sunspot model. The independence assumption of the model residuals is tested by a sensitive diagnostic check. This is accomplished by calculating significance intervals [*McLeod,* 1976*b*] for the estimates of the residual autocorrelation function (RACF). Other diagnostics tests are employed to determine whether the homoscedasticity and normality assumptions are also fulfilled [*Hipel et al.*, 1977*a*; *McLeod*, 1974].

This paper clearly demonstrates for practical applications the exact use of the contemporary model-building methods discussed in an accompanying paper labeled part 1 [*Hipel et al.*, 1977*a*]. However, if necessary the reader is urged to refer to part 1 (in this issue) for an account of the general rules of the application of the modeling procedures or for a theoretical description of the techniques.

In some time series applications the given data are often transformed by a nonlinear transformation such as a Box-Cox transformation. By employing the methods of *Granger and Newbold* [1976] it is now possible to obtain minimum mean square error (mmse) forecasts of the original series when the data have been changed by a nonlinear transformation. The procedure for obtaining mmse forecasts of the untransformed data when there has been a Box-Cox square root transformation is described for the sunspot model. If there has been a natural logarithmic transformation of the data, then the technique of procuring mmse forecasts of the original observations is presented with the airline passenger model.

SAINT LAWRENCE RIVER

Average annual river flows from 1860 to 1957 for the Saint Lawrence River at Ogdensburg, New York, are available from a report by *Yevyevich* [1963]. *Carlson et al.* [1970] fit a (1, 0, 0) model to these data. However, by employing the contemporary approaches to model construction given in part 1 [*Hipel et al.*, 1977*a*] the authors derive a better model for the Saint Lawrence River flows.

In order to identify the form of the model to estimate, the plots of the ACF, PACF, IACF, and IPACF that are drawn in Figures 1–4, respectively, are examined. The ACF does not truncate but rather damps out, suggesting the presence of autoregressive (AR) terms. The 10%, 5%, and 1% significance intervals for the graph of the PACF are for values of the PACF at lags greater than p if the process is $(p, d, 0)$. Notice

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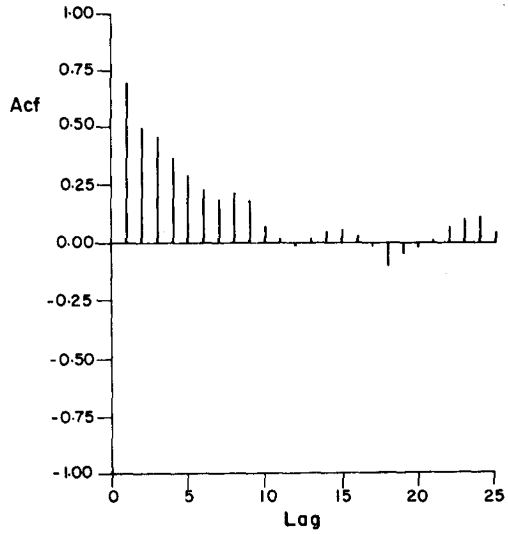

Fig. 1. ACF for the Saint Lawrence River.

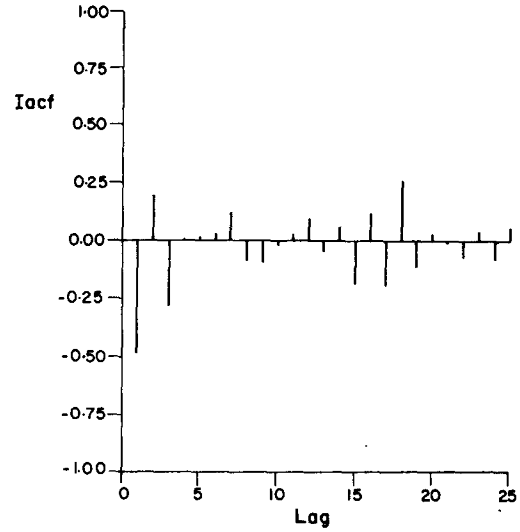

Fig. 3. IACF for the Saint Lawrence River.

that the PACF possesses a significant value at lag 1 and has a value at lag 3 that just touches the 5% significance limit. This effect is more clearly illustrated by the IACF which has definite large values at lags 1 and 3. It may therefore be appropriate to entertain a (3, 0, 0) model with ϕ_2 constrained to zero as a possible process to fit to the Saint Lawrence River data. Although there is a rather large value of the PACF at lag 19 and of the IACF at lag 18, this could be due to chance alone. The IPACF appears to be attenuating rather than truncating. However, for this particular example the ACF definitely damps out, and therefore one would suspect that the IPACF is behaving likewise, thereby indicating the need for AR terms. On the graph for the IPACF (Figure 4) the 10%, 5%, and 1% significance intervals are for values of the IPACF at lags greater than q if the process is $(0, d, q)$.

For the case of the Saint Lawrence River data the IACF most vividly defines the type of model to estimate. However, the remaining three identification graphs reinforce the conclusions drawn from the IACF. Although it is suspected that a (3, 0, 0) model with ϕ_2 constrained to zero should be estimated, a (1, 0, 0) and a (3, 0, 0) model are examined for comparison purposes. *Carlson et al.* [1970] choose a (1, 0, 0) model to estimate for the Saint Lawrence River flows because they only employ the ACF for identification purposes. This illustrates the importance of having many identification procedures so that the proper model is not missed just because one of the identification methods does not explicitly portray the best model to fit.

Table 1 lists the maximum likelihood estimates of the parameter and the standard errors for a (1, 0, 0) model, a (3, 0, 0) model, and a (3, 0, 0) model without the ϕ_2 parameter. The parameter estimates are calculated by using the modified sum of squares technique [*McLeod,* 1976*a*].

Model discrimination can be accomplished by comparing parameter estimates to their standard errors, by using the AIC or by performing the likelihood ratio test. In order to employ the first procedure, first consider the models listed in Table 1. Notice that for both the (3, 0, 0) model and the (3, 0, 0) model without ϕ_2 the estimte $\hat{\phi}_3$ for ϕ_3 is more than twice its standard error. (The circumflex denotes an estimate of the theoretical statistic below it.) Therefore it can be argued that even at the 1% significance level, ϕ_3 is significantly different from

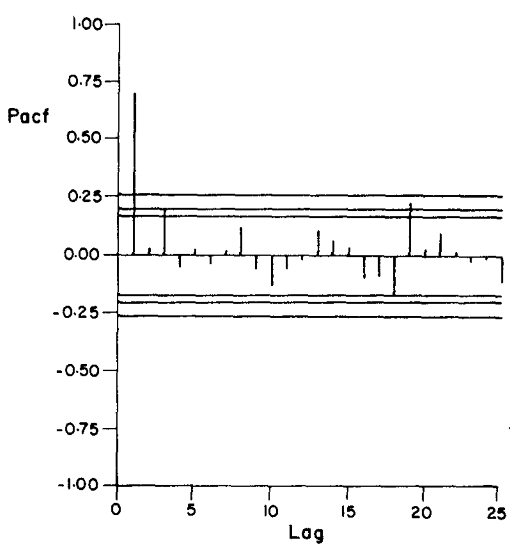

Fig. 2. PACF for the Saint Lawrence River.

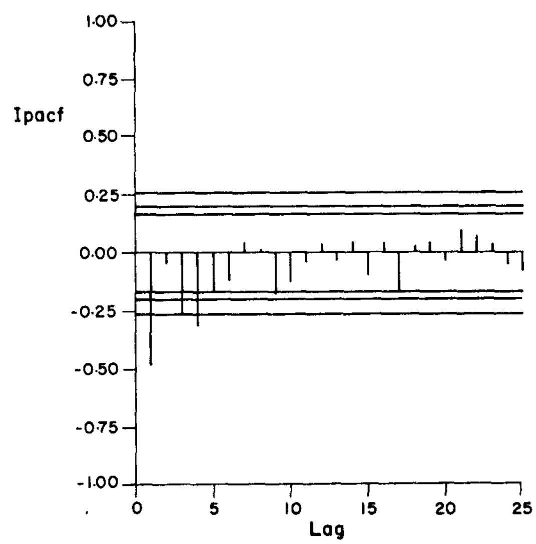

Fig. 4. IPACF for the Saint Lawrence River.

TABLE 1. Parameter Estimates for the Saint Lawrence River Arima Models

Parameter	Maximum Likelihood Estimate	Standard Error
	(1, 0, 0) Model	
ϕ_1	0.7114	0.0714
σ_a	419.66	
	(3, 0, 0) Model	
ϕ_1	0.6584	0.0991
ϕ_2	−0.0863	0.1192
ϕ_3	0.2180	0.0991
σ_a	408.94	
	(3, 0, 0) Model without ϕ_2	
ϕ_1	0.6219	0.0839
ϕ_3	0.1771	0.0840
σ_a	410.15	

zero and should be included in the model. Consequently, the (1, 0, 0) model should not be utilized to model the Saint Lawrence River flows. Furthermore, because the standard error for $\hat{\phi}_2$ for the (3, 0, 0) model is greater than $\hat{\phi}_2$, for model parsimony the (3, 0, 0) model without ϕ_2 is the proper model to select.

When the AIC is employed for model selection, it is not necessary to select subjectively a significance level, as is done in hypothesis testing. By using (11) in the paper by *Hipel et al.* [1977*a*] the AIC for the (1, 0, 0) model, the (3, 0, 0) model, and the (3, 0, 0) model without ϕ_2 are calculated, respectively, as 1176.00, 1175.50, and 1172.07. The (3, 0, 0) model without ϕ_2 has the minimum AIC, and therefore the AIC also indicates that this model should be chosen in preference to the others.

The likelihood ratio test given by *Hipel et al.* [1977*a*] in (13) can be utilized to choose between the (1, 0, 0) model and the (3, 0, 0) model with $\phi_2 = 0$. By substituting $n = 97$, $k = 1$, the residual variance of the (1, 0, 0) model for $\hat{\sigma}_a^2(k)$, $r = 2$, and the residual variance of the (3, 0, 0) model with $\phi_2 = 0$ for $\hat{\sigma}_a^2(r)$, the calculated χ^2 statistic has a magnitude of 4.58. For 1 d.f. this value is significant at the 5% significance level. Therefore this test indicates that the (3, 0, 0) model with $\phi_2 = 0$ should be selected in preference to the (1, 0, 0) model.

The likelihood ratio test can also be employed to test whether a (3, 0, 0) model without ϕ_2 gives as good a fit as the (3, 0, 0) model. Simply substitute into (13) [*Hipel et al.*, 1977*a*], $n = 97$, $k = 2$, the residual variance of the (3, 0, 0) model with $\phi_2 = 0$ for $\hat{\sigma}_a^2(k)$, $r = 3$, and the residual variance of the (3, 0, 0) model for $\hat{\sigma}_a^2(r)$. The calculated χ^2 statistic possesses a value of 0.0569. For 1 d.f. this value is certainly not significant even at the 50% significance level. Therefore the constrained model without ϕ_2 gives an adequate fit and should be used in preference to the (3, 0, 0) model in order to achieve model parsimony. The Arima difference equation for the best model is written as

$$(1 - 0.6219B - 0.1771B^3)(z_t - 6842.25) = a_t \quad (1)$$

where 6842.25 is the mle of the mean of the z_t series.

Diagnostic checks are done to insure that the proper model is selected by checking that assumptions of Arima modeling such as independence, constant variance, and normality of the residual a_t are satisfied. The model in (1) passes all diagnostic checks. The critical assumption of independence can be checked by various methods. However, a sensitive testing procedure is to plot the RACF along with the chosen significance intervals. Figure 5 shows a plot of the RACF for the (3, 0, 0) model with $\phi_2 = 0$. As is the case for all the plots in the application section that possess significance intervals, the 10%, 5%, and 1% intervals are drawn in Figure 5. Although the RACF at lag 18 is rather large, it still lies within the 1% significance interval. This larger value could be due to inherent random variation or to the length of the time series used to estimate it. However, the important values of the RACF for the lower lags all lie well within the 10% significance interval. Therefore the RACF indicates that the chosen model for the Saint Lawrence River satisfies the independence assumption. This fact is also confirmed by the χ^2 distributed portmanteau statistic U_L that is listed in (17) by *Hipel et al.* [1977*a*]. The calculated magnitude of U_L is 13.46 for 18 d.f.

The less important modeling assumptions of homoscedasticity and normality of the residuals are also satisfied. The χ statistic for changes in variance depending on the current level of the series [*Hipel et al.*, 1977*a*] has a magnitude of 0.000081 and a standard error of 0.000341, while the χ statistic for trends in the variance over time [*Hipel et al.*, 1977*a*] possesses a value of 0.002917 with a corresponding standard error of 0.00504. Because in both instances the standard error is greater than the χ statistic, then based upon the information used, it can be assumed that the residuals are homoscedastic. The skewness statistic g_1 for the residuals [*Hipel et al.*, 1977*a*, equation (23)] has a value of −0.1482 and a standard error of 0.3046. The kurtosis statistic g_2 [*Hipel et al.*, 1977*a*, equation (24)] possesses a value of −0.3240 and a standard error of 0.4974. Due to the fact that the standard errors for both g_1 and g_2 are greater than the corresponding statistic, the normality assumption is reasonably well satisfied. Because the residuals are normally distributed and homoscedastic, a Box-Cox transformation of the given data is not required.

The flows used for the Saint Lawrence River are in cubic meters per second. However, if the flows had been in cubic feet per second and a model had been fit to these data, all the AR parameters and standard errors would have been identical with the metric model in (1). Only the mean level of the series and σ_a^2 would be different. In general, no matter what units of measurement are used the AR and the MA parameter estimates and the standard errors will remain the same (for both a nonseasonal and a seasonal model), while the mean level and σ_a^2 will be different.

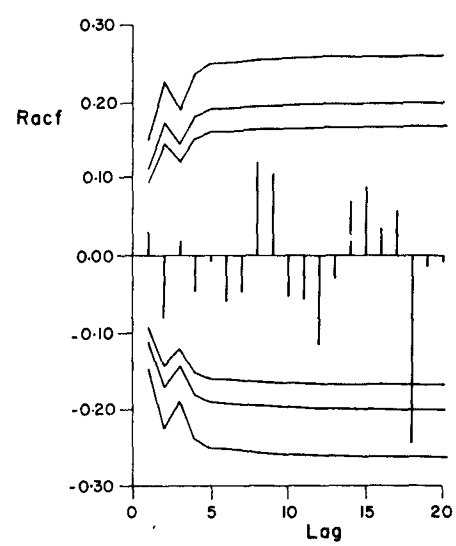

Fig. 5. RACF for the (3, 0, 0) model with $\phi_2 = 0$ for the Saint Lawrence River.

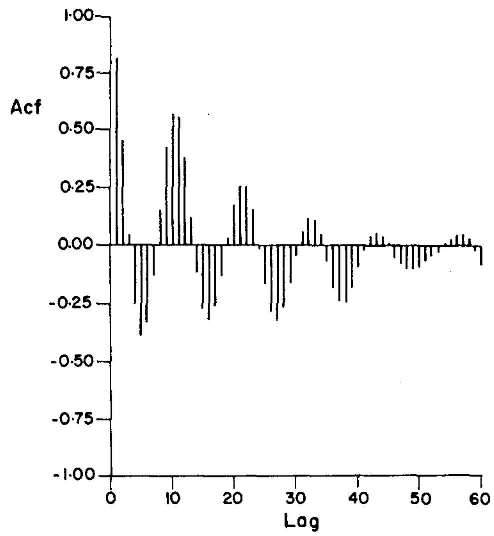

Fig. 6. ACF of the yearly sunspot numbers.

The type of model fit to the Saint Lawrence River data reflects the actual physical situation. The Great Lakes all flow into the Saint Lawrence River, and due to their immense size they are capable of over-year storage. If there is an unusually wet or an unusually dry year, the Great Lakes dampen the effect of extreme precipitation on the flows of the Saint Lawrence River. Because of this the average annual flows are correlated, and the correct model is an AR process rather than white noise.

Annual Sunspot Numbers

The yearly Wolfer sunspot number series is available from 1700 to 1960 in the work of *Waldmeier* [1961]. This series is examined in this paper because of the historical controversies regarding the selection of a suitable model to fit to yearly sunspot numbers and also because sunspot data are of practical importance to geophysicists. Recently, climatologists have discovered that sunspot activity may be important for studying climatic change because of its effect upon global temperature variations [*Schneider and Mass,* 1975]. Also sunspots have long been known to affect the transmission of electromagnetic signals.

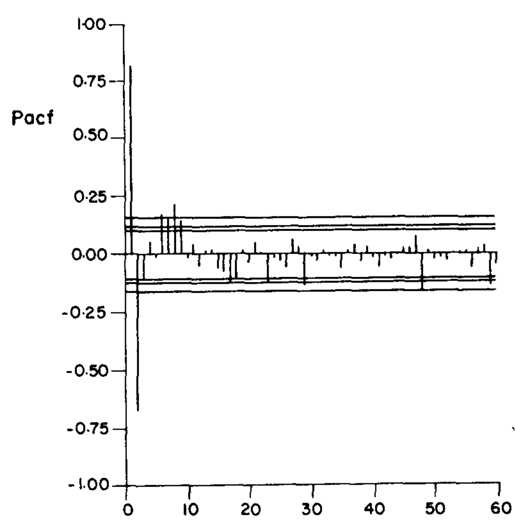

Fig. 7. PACF of the yearly sunspot numbers.

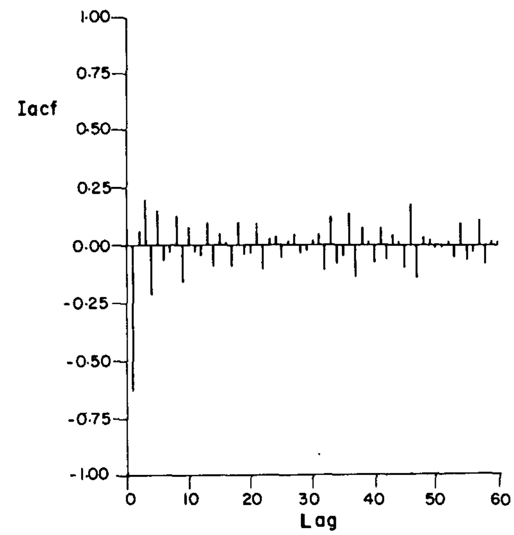

Fig. 8. IACF of the yearly sunspot numbers.

Yule [1927] was the first applied statistician who considered employing an AR process of the order of 2 to model yearly sunspot numbers. *Moran* [1954] examined various types of models for predicting annual sunspot numbers and expressed the need for a better model than an AR(2) process. *Box and Jenkins* [1970, p. 239] fit an AR(3) model to yearly sunspot data. The authors of this paper recommend an AR(9) process with ϕ_3–ϕ_8 constrained to zero to model the annual sunspot series.

Other researchers have determined stochastic sunspot models when the basic time interval is smaller than 1 year. For example, *Whittle* [1954] considered a unit of time of 6 months and developed a bivariate AR scheme to fit to the observed sunspot intensities in the northern and southern solar hemispheres. *Granger* [1957] proposed a special two-parameter curve for the monthly sunspot numbers, but unfortunately, this curve is not useful for forecasting.

The inherent stochastic characteristics of the yearly Wolfer sunspot series complicate the identification of an Arima model to fit to the data. For example, the fact that *Granger* [1957] found that the periodicity of sunspot data follows a uniform distribution with a mean of about 11 years is one reason that historically researchers have had difficulties in modeling sunspot numbers. However, an examination of the graphs of the sunspot ACF, PACF, IACF, and IPACF that are displayed, respectively, in Figures 6–9 does yield some insight into the type of model required to model the sunspot numbers. The ACF follows an attenuating sine wave pattern that reflects the random periodicity of the data and possibly indicates the need for nonseasonal and/or seasonal AR terms in the model. The behavior of the PACF could also signify the need for some type of AR model. In addition to possessing significant values at lags 1 and 2, the PACF also has rather large values at lags 6–9. The IACF has a large magnitude at lag 1, which suggests the importance of a nonseasonal AR lag 1 term in any eventual process that is chosen to estimate. The damping out effect in the first four lags of the IPACF could be a result of a nonseasonal AR component.

When an AR(2) process is fit to the yearly sunspot numbers, the independence, normality, and homoscedasticity assumptions of the residuals are not satisfied. The RACF possesses large values at lags 1, 2, 4, 9, 10, and 11, while the portmanteau statistic U_L has a magnitude of 27.42 for 9 d.f. These facts

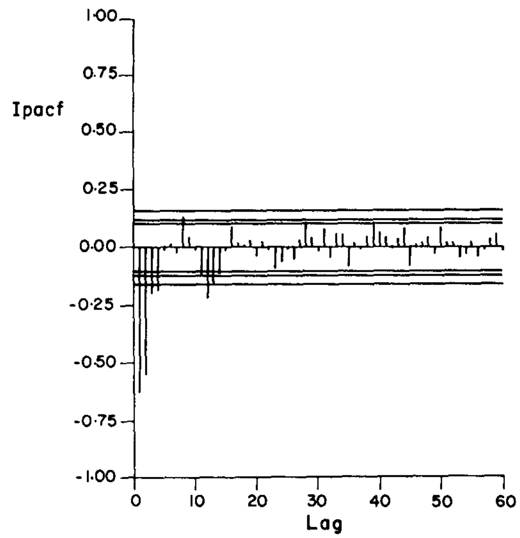

Fig. 9. IPACF of the yearly sunspot numbers.

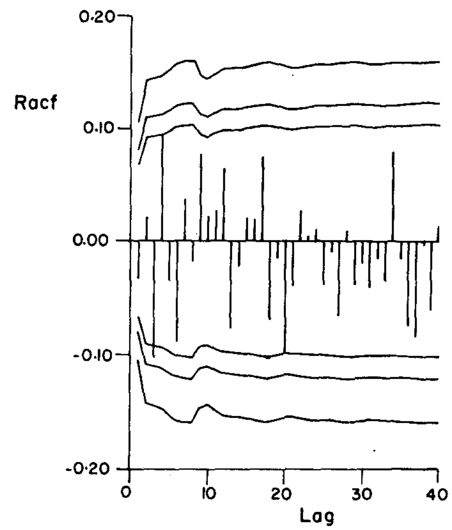

Fig. 10. RACF of the constrained sunspot model.

signify that the residuals are not uncorrelated and that a better model is therefore required. The χ statistic for changes in variance depending on the current level of the series, the χ statistic for trends in the variance over time, the skewness statistic g_1, and the kurtosis statistic g_2 all possess a magnitude that is more than twice their corresponding standard errors. This indicates that a Box-Cox transformation is needed to remove heteroscedasticity and nonnormality of the residuals. By substituting $\lambda = 0.5$ and const = 1.0 into (9) in the work of *Hipel et al.* [1977*a*] a Box-Cox transformation causes the residuals to be homoscedastic and approximately normally distributed. The parameter constant is set equal to 1 because there are some zero values in the sunspot time series. A data transformation usually cannot correct residual correlation, and a different type of model than an AR(2) process is therefore required to rectify the situation.

If an AR(3) model with $\lambda = 0.5$ and const = 1.0 is estimated, the $\hat{\phi}_3$ parameter has a magnitude of -0.1032 and a standard error of 0.0616. Because $\hat{\phi}_3$ is less than twice its standard error, for the sake of model parsimony it should not be incorporated into the model. Note that *Box and Jenkins* [1970, p. 239, Table 7.13] obtain a parameter estimate for ϕ_3 that is just slightly more than twice its standard error. However, Box and Jenkins do not employ a data transformation to remove heteroscedasticity and nonnormality and only use the Wolfer sunspot series from 1770 to 1869.

The RACF for the residual estimates of the AR(3) model possesses a large value at lag 9. This fact implies that it may be advisable to estimate an AR(9) process with ϕ_3–ϕ_8 constrained to zero. The estimates and standard errors for this model are listed in Table 2, while the difference equation is written in (2):

$$(1 - 1.2434B + 0.5192B^2 - 0.1954B^9) \cdot (z_t^{(\lambda)} - 11.77) = a_t \qquad (2)$$

where $z_t^{(\lambda)} = (1/0.5)[(z_t + 1.0)^{0.5} - 1.0]$ and 11.77 is the mle of the mean of the $z_t^{(\lambda)}$ series for the constrained AR(9) model.

The model in (2) satisfies all the modeling assumptions of the residuals. A plot of the RACF in Figure 10 shows that the residuals are uncorrelated. All of the estimated values of the RACF fall within the 5% significance interval. The χ^2 distributed portmanteau statistic U_L has a value of 18.85 for 22 d.f. Therefore the U_L statistic also confirms that the residuals are not correlated. The diagnostic checks for homoscedasticity and normality of the residuals reveal that these assumptions are also fulfilled. The model in (2) therefore adequately models the yearly Wolfer sunspot numbers. Other types of constained models were examined, but the AR(9) process with ϕ_3–ϕ_8 constrained to zero is the only model that was found to be satisfactory.

An alternative approach to modeling the sunspot series by a constrained model is to consider a multiplicative Box-Jenkins seasonal process. The large values of the PACF at lags 1 and 2 in Figure 7 indicate the need for the nonseasonal AR parameters ϕ_1 and ϕ_2. The behavior of the ACF in Figure 6 could be due to a seasonal AR component. On the basis of these facts and other previously mentioned information it is appropriate to estimate the parameters of a $(2, 0, 0) \times (1, 0, 0)_9$ model. The parameter estimates and standard errors for this seasonal model with $\lambda = 0.5$ and const = 1.0 are given in Table 3. The multiplicative seasonal Arima difference equation is written in (3):

TABLE 2. Parameter Estimates for the Constrained Sunspot Model

Parameter	Maximum Likelihood Estimate	Standard Error
ϕ_1	1.2434	0.0470
ϕ_2	−0.5192	0.0458
ϕ_9	0.1954	0.0249
σ_a	2.0569	

TABLE 3. Parameter Estimates for the Seasonal Sunspot Model

Parameter	Maximum Likelihood Estimate	Standard Error
ϕ_1	1.3783	0.0456
ϕ_2	−0.6770	0.0460
Φ_1	0.2142	0.0616
σ_a	2.1958	

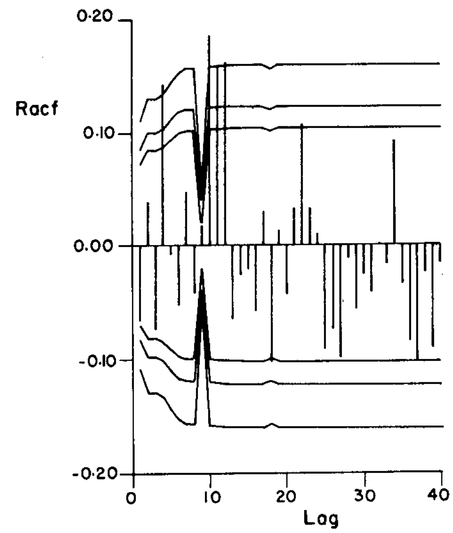

Fig. 11. RACF of the seasonal sunspot model.

$$(1 - 1.3783B + 0.6770B^2)(1 - 0.2142B^9) \cdot (z_t^{(\lambda)} - 10.62) = a_t \quad (3)$$

where $z_t^{(\lambda)} = (1/0.5)[(z_t + 1.0)^{0.5} - 1.0]$ and 10.62 is the mle of the mean of the $z_t^{(\lambda)}$ series for the seasonal model.

Unfortunately, the seasonal model in (3) does not fare well with the whiteness checks. As is shown in Figure 11, the RACF possesses a large value at lag 10 that lies well outside the 1% significance interval. The RACF estimates at lags 4 and 12 are also outside the 1% significance interval, while the RACF at lag 11 just touches the 1% significance limit. The U_L statistic has a value of 43.35 for 22 d.f. This fact also indicates that the seasonal model is inadequate to model properly the sunspot data. Other diagnostic checks reveal that the homoscedasticity and normality assumptions are reasonably well satisfied. Although the authors examined other types of multiplicative seasonal Arima models, the process in (3) could not be improved upon by another multiplicative seasonal model.

From the foregoing discussion regarding a process to model the sunspot data it is evident that the constrained AR(9) process gives the best fit. The AIC can also be employed to verify which is the best model to select. The AIC for the AR(2), the AR(3), the seasonal Arima $(2, 0, 0) \times (1, 0, 0)_9$, and the constrained AR(9) model without ϕ_3–ϕ_8 are calculated to be, respectively, 1332.02, 1331.30, 1323.51, and 1288.63. For all four types of models the data are transformed by setting $\lambda = 0.5$ and const = 1.0. The constrained AR(9) process has the minimum AIC, and therefore according to the AIC decision-making procedure this model should be selected in preference to the other three. Because the constrained AR(9) model is chosen in preference to a seasonal model, it is also referred to as a nonmultiplicative model.

An important application of Arima models in water resources is forecasting. *Box and Jenkins* [1970, chapter 5] describe how to obtain mmse forecasts for Arima models if the original data have not been transformed by a nonlinear transformation. The same technique also applies for forecasting transformed values if the Arima model has been fit to transformed data. Recently, *Granger and Newbold* [1976] discussed a mmse method of forecasting the original series when there has also been a nonlinear transformation.

Let $z_t^{(\lambda)}(k)$ denote a mmse forecast at origin t for lead time k for the transformed series. The exponent of $z_t^{(\lambda)}(k)$ could, for example, indicate some type of Box-Cox transformation. A naive approach to obtain a forecast for the original untransformed series would be to take the inverse transformation of $z_t^{(\lambda)}(k)$. However, *Granger and Newbold* [1976] describe an optimal procedure for procuring a forecast $z_t(k)$ at origin t for lead time k for the untransformed series.

As an illustrative example, consider the sunspot series model where there is a Box-Cox transformation with $\lambda = 0.5$ and const = 1.0. This transformation is written as

$$z_t^{(\lambda)} = (1.0/\lambda)[(z_t + \text{const})^\lambda - 1.0] \quad (4)$$

where $\lambda = 0.5$ and const = 1.0. *Granger and Newbold* [1976, p. 197] give the mmse forecast of the untransformed series as

$$z_t(k) = [0.5z_t^{(\lambda)}(k) + 1.0]^2 - [1.0 + \text{Var}(k)] \quad (5)$$

where $z_t(k)$ is the mmse forecast of the original series, $z_t^{\lambda}(k)$ is the mmse forecast of the $z_t^{(\lambda)}$ series obtained by using the methods of *Box and Jenkins* [1970, chapter 5], and Var (k) is the variance of the forecast error of $z_t^{\lambda}(k)$ [*Box and Jenkins*, 1970, p. 128, equation (5.1.16)].

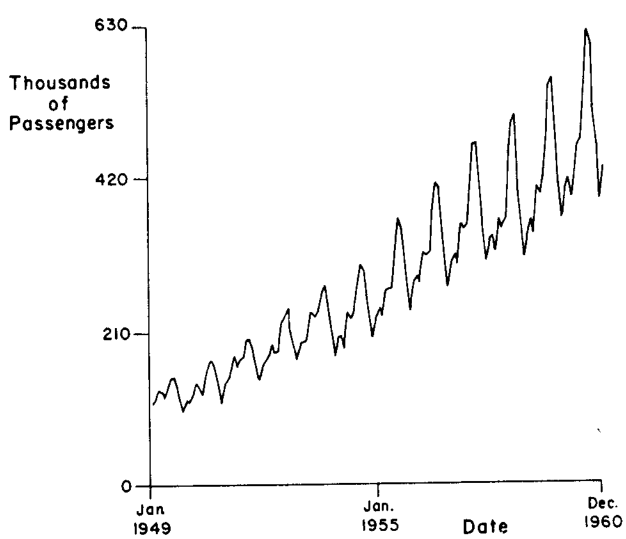

Fig. 12. Total monthly international airline passenger data.

Airline Passenger Data

Total monthly international airline passenger data from 1949 to 1960 are listed by *Box and Jenkins* [1970, p. 531]. Although Box and Jenkins determine a model for the airline passenger data, the same model is reformulated in this paper to demonstrate the usefulness of the contemporary methods described by *Hipel et al.* [1977*a*] for seasonal model construction.

A plot of the airline passenger data in Figure 12 reveals important information about the observations. The periodic peaks in the data reflect the seasonality of the observations. The series is seasonal due to a high travel period during the summer months and a lesser peak travel time in the spring. It is obvious from the general magnitude increase of all the data with time that there is a linear trend component present. In some instances a physical understanding of the process being analyzed allows for the incorporation of deterministic components into the model to account for seasonality and/or trends. For instance, seasonality may be modeled by a Fourier

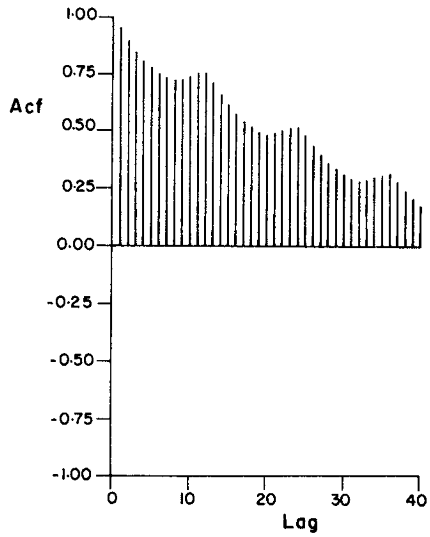

Fig. 13. ACF of the logarithmic airline data.

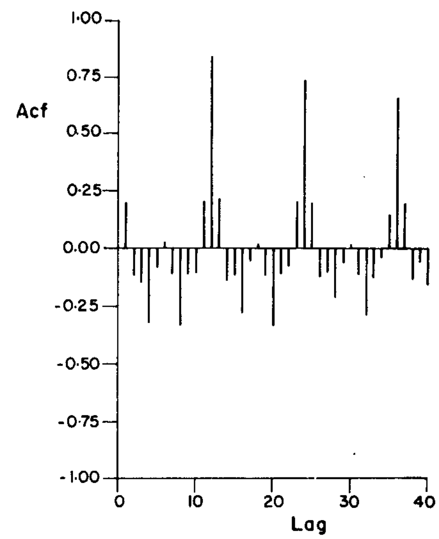

Fig. 15. ACF of the nonseasonally differenced logarithmic airline data.

series, while trend might be accounted for by a polynomial. However, for the airline passenger data a pure stochastic seasonal Arima model is fit to the data. This model stochastically accounts for the inherent properties of the given data.

Another important characteristic of the raw data is that the variance increases with time. This fact is detected by noticing the escalating amplitude of the seasonal wave pattern. A change in variance over time of the original data would eventually be mirrored by heteroscedasticity in the residuals of the model fit to the data. To rectify the situation from the start, natural logarithms are taken of the data.

The ACF of the logarithmic airline data that are plotted in Figure 13 demonstrates that the logarithmic data are seasonal due to the pronounced peaks of the ACF at lags that are multiples of 12. Because the ACF attenuates very slowly, this indicates the need for seasonal and/or nonseasonal differencing. Figure 14 illustrates that seasonal differencing removes the seasonal wave pattern in the ACF but fails to cause the ACF to damp out more rapidly than it does in Figure 13. When calculating the ACF in Figure 14, or in general when computing the ACF of any series that has been differenced, the mean of the w_t series is not removed. This procedure precludes missing any deterministic component that may still be present even after differencing. As is demonstrated in Figure 15, nonseasonal differencing of the logarithmic airline data fails to remove large but slowly decaying values of the ACF at lags that are multiples of 12. Therefore it is appropriate to difference the logarithms of the observations both seasonally and nonseasonally. Figure 16 illustrates that no further differencing of the logarithmic data is required because the ACF appears to more or less truncate after lag 12.

In order to identify the number of AR and MA terms required in the model of the nonseasonally and the seasonally differenced logarithmic airline data, the graphs of the ACF, the PACF, the IACF, and the IPACF that are shown in Figures 17–19, respectively, are interpreted simultaneously. Notice that both the ACF and the IPACF have significant values at lags 1 and 12. This indicates that possibly both a nonseasonal and a seasonal MA term are required. The IACF plot reinforces these conclusions.There are rather large values

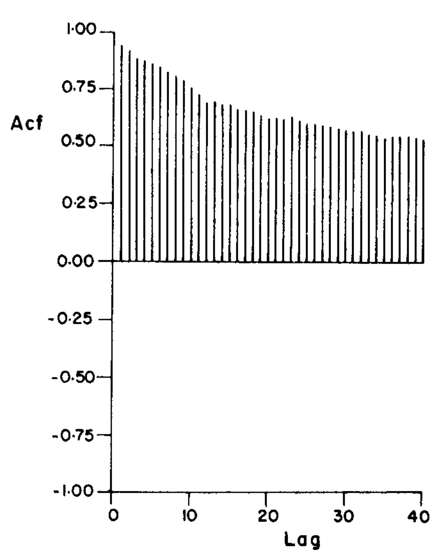

Fig. 14. ACF of the seasonally differenced logarithmic airline data.

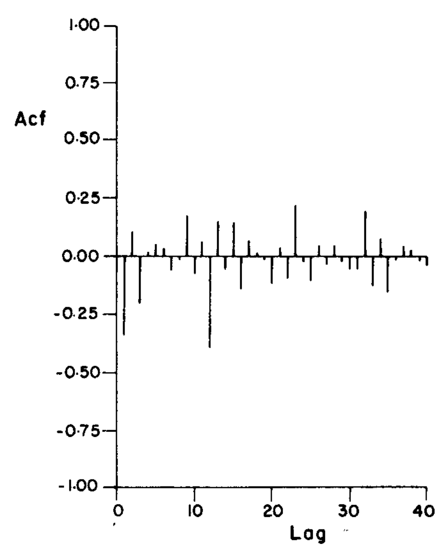

Fig. 16. ACF of the nonseasonally and seasonally differenced airline data.

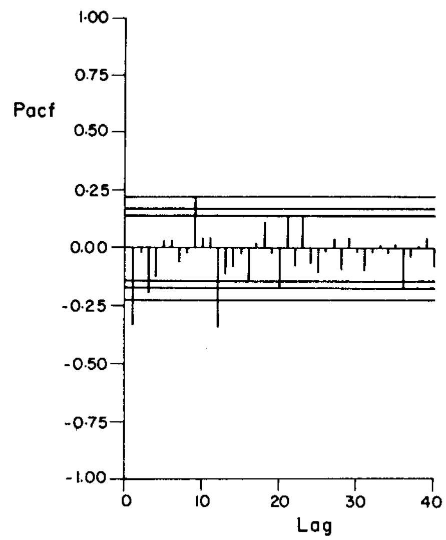

Fig. 17. PACF of the differenced logarithmic airline data.

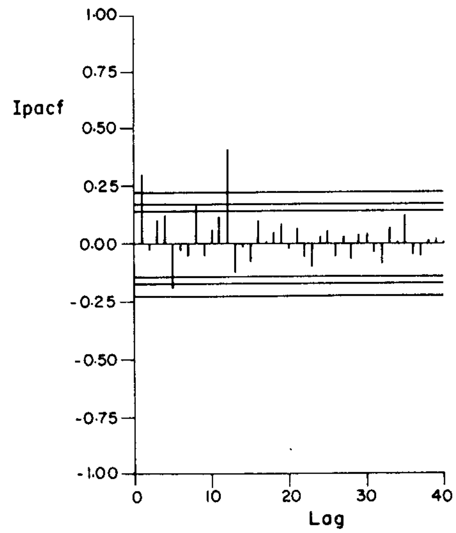

Fig. 19. IPACF of the differenced logarithmic airline data.

of the IACF for the first four lags and also some large values just after lags that are multiples of 12. This implies the presence of a nonseasonal MA component. Because the IACF attenuates at lags that are multiples of 12, this fact implies that a seasonal MA term is also needed. A possible process to try for modeling the logarithmic airline data is therefore a $(0, 1, 1) \times (0, 1, 1)_{12}$ model.

The plot of the PACF does not delineate as clearly as the other identification graphs the type of model to estimate. Because of the significant values of the PACF at lags 1 and 12 it seems that both an AR nonseasonal and an AR seasonal term are required. However, these conclusions contradict previous results. A closer inspection of the PACF reveals that some larger values at lags 3 and 4 could suggest the need for a nonseasonal MA term. There is not a significant value of the PACF at lag 24, but a rather large value at lag 36 may imply an attenuating effect due to a seasonal MA component. For this particular example the PACF contributes little to the identification and actually adds some confusion concerning which model to estimate. However, when all four graphs are examined as a unit, it is obvious that the $(0, 1, 1) \times (0, 1, 1)_{12}$ model is the proper process to estimate. When considering other time series applications the PACF usually enhances the identification procedure.

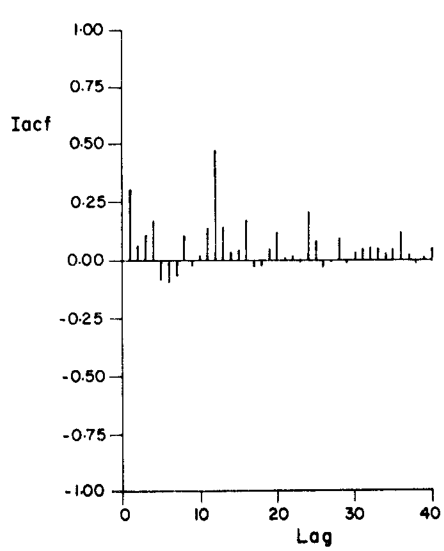

Fig. 18. IACF of the differenced logarithmic airline data.

In practice, the authors have discovered that the utilization of all the identification procedures used in this paper greatly simplifies model identification, especially for seasonal Arima processes. Unfortunately, *Box and Jenkins* [1970, chapter 9] do not explain explicitly how to use the PACF for seasonal model identification. This could be the reason that *McMichael and Hunter* [1972] and *McKerchar and Delleur* [1974] fail to employ the PACF for seasonal Arima model identification and therefore appear to have some difficulty in determining which models to estimate.

Table 4 lists the parameter estimates and standard errors for a $(0, 1, 1) \times (0, 1, 1)_{12}$ model fit to the natural logarithms of the total monthly international airline passenger data. Estimates are calculated by using both the modified sum of squares technique [*McLeod*, 1976*a*] and the unconditional sum of squares method [*Box and Jenkins*, 1970, chapter 7]. Notice that the better estimates when using the modified sum of squares method do differ somewhat from the other estimates even in apparently large samples. In particular, the two estimates for Θ_1 differ by approximately 80% of their standard errors. Furthermore, the new estimation procedure involves only a slight increase in computations in comparison with the old method.

The Arima difference equation for the airline model is written as

$$(1 - B)(1 - B^{12}) \ln z_t = (1 - 0.4018B) \cdot (1 - 0.5569B^{12})a_t \qquad (6)$$

TABLE 4. Parameter Estimates for the Airline Model

	Modified Sum of Squares Method		Unconditional Sum of Squares Method	
Parameter	mle*	SE†	mle*	SE†
θ_1	0.4018	0.0800	0.3959	0.0802
Θ_1	0.5569	0.0726	0.6135	0.0690
σ_a	0.03672		0.03664	

*The mle stands for maximum likelihood estimate.
†SE stands for standard error.

A plot of the RACF in Figure 20 for the model in (6) shows that the independence assumption of the residuals is satisfied. All the values of the RACF lie within the 10% significance interval except for the RACF estimate at lag 23, which is still inside the 1% significance interval. The χ^2 distributed U_L statistic has a value of 21.45 for 22 d.f. and therefore also indicates that the residuals possess no significant nonwhiteness. Other diagnostic checks reveal that both the homoscedasticity and the normality assumption for the residuals are fulfilled. Therefore on the basis of the information used, the chosen seasonal Arima model adequately models the airline data.

Although a seasonal $(0, 1, 1) \times (0, 1, 1)_{12}$ model is determined for the logarithmic airline data in this paper, in other situations it may be advisable to design a model that avoids differencing. When a model contains differencing this means that the process is nonstationary, and therefore if the model is used for simulation purposes, the generated data are not restricted to any mean level. In engineering applications such as reservoir design, stationarity of monthly average river flows is essential in order to avoid massive overdesign. To overcome this problem, researchers often incorporate a deterministic component, or transformation, into the model to remove seasonality. Then a nonseasonal autoregressive moving average (Arma) model is fit to the remaining stochastic component. For example, *McMichael and Hunter* [1972] describe how to eliminate seasonality for daily water temperature and discharge data by employing a Fourier series and other periodic polynomials for the deterministic component. *McKerchar and Delleur* [1974] standardize the data for each month to eliminate seasonality in average monthly river flow observations. *Hipel* [1975] discusses standardization and other methods of removing seasonality and then explains the proper procedure for determining an Arma model for the stochastic component. *Tao and Delleur* [1976] describe harmonic and standardization procedures for removing the seasonal component from time series.

Forecasting is an important application of airline passenger data. In many types of time series applications it is often appropriate to compute the transformation given by

$$z_t^{(\lambda)} = \ln (z_t + \text{const}) \tag{7}$$

where $z_t^{(\lambda)}$ is the transformed z_t series and const is a constant that is chosen large enough to make all values of the $z_t^{(\lambda)}$ series positive. For the case of the airline passenger model, the constant is given a value of zero because no zero or negative values are present in the original series.

When the original z_t series is changed by the transformation in (7), the mmse forecast of the untransformed series at origin t for lead time k is [*Granger and Newbold,* 1976, p. 197]

$$z_t(k) = \exp \{z_t^{(\lambda)}(k) + [\text{Var } (k)/2]\} - \text{const} \tag{8}$$

where $z_t(k)$ is the mmse forecast of the original series for the Arima model fit to the data, $z_t^{(\lambda)}(k)$ is the mmse of the $z_t^{(\lambda)}$ series obtained by using the approach described by *Box and Jenkins* [1970, chapter 5], and Var (k) is the variance of the forecast error of $z_t^{(\lambda)}(k)$ [*Box and Jenkins,* 1970, p. 128, equation (5.1.16)]. In order to derive (8) it is assumed that the $z_t^{(\lambda)}(k)$ forecast is approximately normally distributed. Note that the normality assumption is not invoked to obtain (4).

If the model in (6) for the airline passenger data is employed for forecasting, the first step is to obtain mmse forecasts for $z_t^{(\lambda)} = \ln z_t$ by using the methods given by *Box and Jenkins* [1970, pp. 306–313]. Then (8) is utilized to determine mmse forecasts for the original untransformed z_t series.

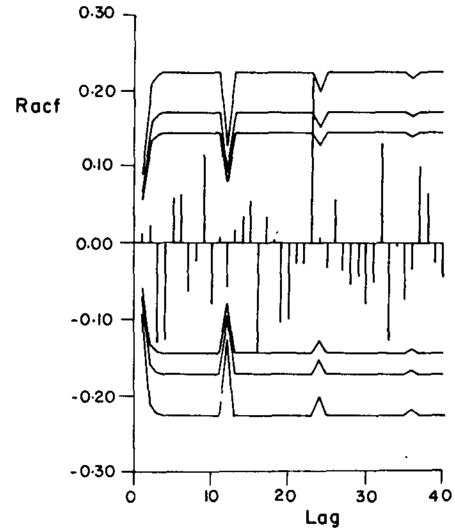

Fig. 20. RACF of the seasonal model for the airline data.

Conclusions

This paper demonstrates that in practical applications, improved Arima models can be obtained readily by using the recent modeling techniques that are described by *Hipel et al.* [1977*a*]. Although more methods are now available for use at all three stages of model construction, this does not complicate the modeling process. Rather, the model-building procedure is simplified and at the same time strengthened. Utilization of more types of identification plots allows for a fairly rapid convergence to a tentative model to estimate. At the estimation stage, better parameter estimates can be calculated by employing the modified sum of squares technique [*McLeod,* 1976*a*]. Rigorous diagnostic checks insure that an appropriate Arima model is ultimately selected, and therefore the user should have more confidence in the chosen Arima process. If problems arise due to heteroscedasticity and/or nonnormality of the residuals, these shortcomings can often be overcome by a Box-Cox transformation of the data. Finally, model parsimony can be insured by utilization of techniques such as the AIC or the likelihood ratio test.

For the case of the Saint Lawrence River model in (1) a better model is obtained here than was obtained by *Carlson et al.* [1970]. This is due to the fact that in addition to the ACF being examined, the PACF, the IACF, and the IPACF are also studied at the identification stage. The AIC and other checks confirm that a (3, 0, 0) model with ϕ_2 constrained to zero is indeed the proper model to select. Model parsimony is preserved, while simultaneously the assumptions of the residuals are not violated.

An AR(9) model without ϕ_3–ϕ_8 adequately models the annual sunspot numbers. A check for homoscedasticity of the residuals reveals that it is necessary to employ a Box-Cox transformation to remove heteroscedasticity. A square root transformation with the parameter $\lambda = 0.5$ and const $= 1.0$ rectifies the situation. Diagnostic checks and also the AIC indicate that the constrained AR(9) process should be selected in preference to an AR(2), an AR(3), or a multiplicative seasonal Box-Jenkins model.

The authors agree with the findings of *Box and Jenkins*

[1970, chapter 9] that a (0, 1, 1) × (0, 1, 1)$_{12}$ model is the proper seasonal multiplicative process to fit to the natural logarithms of the monthly airline passenger data. However, it is clearly shown in this paper how all the identification plots can be utilized to determine a tentative seasonal model to estimate. Furthermore, as is shown in Table 4, the modified sum of squares estimation method gives better estimates for the moving average parameters than the estimates obtained from the unconditional sum of squares technique.

If the original data are changed by a nonlinear transformation, it is now possible to procure mmse forecasts of the untransformed series [*Granger and Newbold,* 1976]. This forecasting procedure is described for both a square root and a natural logarithmic transformation.

Aside from determining nonseasonal or seasonal Arima models for given data sets the contemporary model-building procedures given by *Hipel et al.* [1977*a*] can be extended for use in other types of applications. An important consideration in water resources is how to model the effects of either man-induced or natural interventions on natural hydrologic variables. For example, what are the stochastic alterations of a newly constructed reservoir on the mean level of the downstream flow regime? *Hipel et al.* [1975, 1977*b*] determine intervention models for water resource applications by employing similar modeling procedures to those used in this paper.

NOTATION

- a_t white noise time series.
- AR(p) autoregressive process of order p.
- B backward shift operator.
- const additive constant for a Box-Cox transformation.
- d order of the nonseasonal differencing operator.
- D order of the seasonal differencing operator.
- g_1 estimated residual skewness.
- g_2 estimated residual kurtosis.
- ln natural logarithm.
- mle maximum likelihood estimate.
- mmse minimum mean square error estimate.
- p order of the nonseasonal AR operator.
- (p, d, q) nonseasonal Arima model.
- $(p, d, q) \times (P, D, Q)_s$ seasonal Box-Jenkins Arima model.
- P order of the seasonal AR operator.
- q order of the nonseasonal MA operator.
- Q order of the seasonal MA operator.
- s seasonal length.
- t discrete time.
- U_L portmanteau statistic calculated from the estimates of the RACF up to lag L.
- Var (k) variance of the forecast error of $z_t^{(\lambda)}(k)$.
- z_t discrete time series value at time t.
- $z_t^{(\lambda)}$ transformation of z_t.
- $z_t(k)$ mmse forecast of z_t at origin t for lead time k.
- $z_t^{(\lambda)}(k)$ mmse forecast of $z_t^{(\lambda)}$ at origin t for lead time k.
- θ_i ith nonseasonal MA parameter.
- Θ_i ith seasonal MA parameter.
- λ exponent of a Box-Cox transformation.
- σ_a^2 variance of a_t.
- $\sigma_a^2(k)$ residual variance of a process with k parameters.
- $\sigma_a^2(r)$ residual variance for an AR process with r parameters.
- ϕ_i ith nonseasonal AR parameter.
- Φ_i ith seasonal AR parameter.
- χ constant to be estimated in the homoscedasticity test for the residuals.
- χ^2 chi-squared random variable.

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