HYPERBOLIC DECAY TIME SERIES

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Abstract. Hyperbolic decay time series such as fractional Gaussian noise or fractional autoregressive moving-average processes exhibit two distinct types of behaviour: strong persistence or antipersistence. Beran (*Statistics for Long Memory Processes*. London: Chapman and Hall, 1994) characterized the family of strongly persistent time series. A more general family of hyperbolic decay time series is introduced and its basic properties are characterized in terms of the autocovariance and spectral density functions. The random shock and inverted form representations are derived. It is shown that every strongly persistent series is the dual of an antipersistent series and vice versa. The asymptotic generalized variance of hyperbolic decay time series with unit innovation variance is shown to be infinite which implies that the variance of the minimum mean-square-error one-step linear predictor using the last k observations decays slowly to the innovation variance as k gets large.

Keywords. Covariance determinant; duality in time series; fractional differencing and fractional Gaussian noise; long-range dependence; minimum mean-square-error predictor; non-stationary time series modelling.

1. INTRODUCTION

Let Z_t, $t = 1, 2, \ldots$, denote a covariance-stationary, purely non-deterministic time series with mean zero and with autocovariance function $\gamma_Z(k) = \operatorname{cov}(Z_t, Z_{t-k})$. As is discussed by Beran (1994), many long-memory processes such as the fractional Gaussian noise (FGN) process (Mandelbrot, 1983) and the fractional autoregressive moving-average (FARMA) process (Granger and Joyeux, 1980; Hosking, 1981) may be characterized by the property that $k^{\alpha}\gamma_Z(k) \to c_\gamma$ as $k \to \infty$, for some $\alpha \in (0, 1)$ and $c_\gamma > 0$. Equivalently,

$$\gamma_Z(k) \sim c_\gamma k^{-\alpha}. \tag{1}$$

As noted by Box and Jenkins (1976), the usual stationary ARMA models, on the other hand, are exponentially damped since $\gamma_Z(k) = \mathrm{O}(r^k)$, $r \in (0, 1)$.

Beran (1994 p. 42) shows that an equivalent characterization of strongly persistent time series is

$$f_Z(\lambda) \sim c_f \lambda^{\alpha-1} \qquad \text{as } \lambda \to 0 \tag{2}$$

where $\alpha \in (0, 1)$, $c_f > 0$ and $f_Z(\lambda)$ is the spectral density function given by $f_Z(\lambda) = \sum \gamma_Z(k)\, e^{-ik\lambda}/(2\pi)$. Theorem 1 below summarizes some results stated without proof in Beran (1994, Lemma 5.1). Since not all time series satisfying Equations (1) or (2) are invertible, the restriction to invertible processes is required.

THEOREM 1. *The time series Z_t satisfying (1) or (2) may be written in random shock form as $Z_t = A_t + \sum \psi_l A_{t-l}$, where $\psi_l \sim c_\psi l^{-(1+\alpha)/2}$, $c_\psi > 0$, and A_t is white noise. Assuming that Z_t is invertible, the inverted form may be written $Z_t = A_t + \sum \pi_l Z_{t-l}$, where $\pi_l \sim c_\pi l^{-(3-\alpha)/2}$, $c_\pi > 0$, and A_t is white noise.*

PROOF. By the Wold decomposition, any purely non-deterministic time series may be written in random shock form. Now assume the random shock coefficients specified in the theorem and we will derive (1). Assuming $\operatorname{var}(A_t) = 1$, $\gamma_Z(k) = \psi_k + \sum \psi_h \psi_{h+k}$,

$$\gamma_Z(k) \sim \psi_k + c_\psi^2 \sum_{h=1}^{\infty} h^{-(1+\alpha)/2}(h+k)^{-(1+\alpha)/2}$$

$$\sim \psi_k + c_\psi^2 \int_1^\infty h^{-(1+\alpha)/2}(h+k)^{-(1+\alpha)/2}\, dh + R_k$$

where the Euler summation formula (Graham *et al.*, 1989, 9.78, 9.80) is used in the last step and

$$R_k = \left\{ -\frac{1}{2} F(h) + \frac{1}{12} F'(h) + \frac{\theta}{720} F'''(h) \right\}\Bigg|_1^\infty$$

where $\theta \in (0, 1)$ and $F(h) = h^{-(1+\alpha)/2}(h+k)^{-(1+\alpha)/2}$. It is easily shown that $k^\alpha R_k \to 0$ as $k \to \infty$. Hence,

$$\gamma_Z(k) \sim \psi_k + c_\psi^2 \int_1^\infty h^{-(1+\alpha)/2}(h+k)^{-(1+\alpha)/2}\, dh$$

$$\sim \psi_k + k^{-\alpha} c_\psi^2 \int_{1/k}^\infty x^{-\beta}(x+1)^{-\beta}\, dx$$

where $\beta = (1+\alpha)/2$. Using Mathematica,

$$\int_0^\infty x^{-\beta}(x+1)^{-\beta}\, dx = \frac{2^{2\beta}\Gamma(1-\beta)\Gamma(-\frac{1}{2}+\beta)}{4\sqrt{\pi}}$$

and so (1) now follows with $c_\gamma = c_\psi^2 2^{\alpha-1}\Gamma\{(1-\alpha)/2\}\Gamma(\alpha/2)/\sqrt{\pi}$, where $\Gamma(\cdot)$ is the gamma function. This shows that ψ_k is a possible factorization of γ_k and this suffices to establish that $Z_t = A_t + \sum \psi_l A_{t-l}$.

For any stationary invertible linear process Z_t,

$$\gamma_Z(k) = \sum_{h=1}^{\infty} \pi_h \gamma_Z(k-h). \tag{3}$$

Assume that $\gamma_Z(k)$ satisfies Equation (1) and that $\pi_l \sim c_\pi l^{-(3-\alpha)/2}$; then we will show that Equation (3) is satisfied.

$$\gamma_Z(k) \sim \gamma_Z(0)\pi_k + c\sum_{h=1}^{k-1} h^{-3/2+\alpha/2}(k-h)^{-\alpha} + c\sum_{h=k+1}^{\infty} h^{-3/2+\alpha/2}(h+k)^{-\alpha}$$

where $c = c_\pi c_\gamma$. Now $\gamma_Z(0)\pi_k/\gamma_Z(k) \sim 0$ so the first term will drop out. In the second term, for $k \gg h$, $(k-h)^{-\alpha} \sim k^{-\alpha}$ and

$$\sum_{h=1}^{k-1} h^{-3/2+\alpha/2} \sim H_\beta \qquad \text{as } k \to \infty$$

where $H_\beta = \sum_{h=1}^{\infty} h^{-\beta} < \infty$, $\beta = 3/2 - \alpha/2$. In the final term, when $h \gg k$, $(h+k)^{-\alpha} \sim h^{-\alpha}$, and so

$$\begin{aligned}\sum_{h=k+1}^{\infty} h^{-3/2+\alpha/2}(h+k)^{-\alpha} &\sim \sum_{h=k+1}^{\infty} h^{-3/2-\alpha/2}\\ &\sim \int_{k+1}^{\infty} h^{-3/2-\alpha/2}\,dh\\ &\sim (k+1)^{-(1+\alpha)/2}.\end{aligned}$$

Again the Euler summation formula is used in the last step. Thus the final term is smaller asymptotically than γ_k. This establishes the asymptotic equivalence of the left-hand side and the right-hand side of Equation (3) and the theorem since $\gamma_Z(k)$ uniquely determines the coefficients π_l in the inverted model. ■

The FARMA model of order (p, q) (Granger and Joyeux, 1980; Hosking, 1981) may be defined by the equation

$$\phi(B)(1-B)^d Z_t = \theta(B)A_t \tag{4}$$

where $|d| < 0.5$, A_t is white noise with variance σ_A^2, $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$, and $\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$. For stationarity and invertibility it is assumed that all roots of $\phi(B)\theta(B) = 0$ are outside the unit circle and $|d| < 0.5$. The series is strongly persistent or antipersistent according as $0 < d < 0.5$ or $-0.5 < d < 0$. The special case where $p = q = 0$ is known as fractionally differenced white noise.

Antipersistent series may arise in practice when non-stationary time series are modelled. As suggested by Box and Jenkins (1976) a non-stationary time series can often be made stationary by differencing the series until stationarity is reached. Sometimes the resulting stationary time series may be usefully modelled by an antipersistent form of the FARMA model. An illustrative example is provided by the annual US electricity consumption data for 1920–70. Hipel and McLeod (1994, pp. 154–59) modelled the square-root consumption using an ARIMA (0, 2, 1) but a better fit is obtained by modelling the second differences of the square-root consumption as fractionally differenced white noise with $d = -0.4477 \pm 0.1522$ SD. The Akaike information criterion (AIC) for the latter model is 1011.5, compared with 1020.4. Diebold and Rudebusch (1989) and Beran (1995) also used this approach for modelling non-stationary data.

The determinant of the covariance matrix of n successive observations Z_t, $t = 1, \ldots, n$, is denoted by $G_Z(n) = \det\{\gamma_Z(i-j)\}$. It will now be shown in Theorem 2 that, for fractionally differenced white noise, $g_Z(n) = \sigma_A^{-2n} G_Z(n) \to \infty$ as $n \to \infty$, where $0 < \sigma_A^2 < \infty$ is the innovation variance given by Kolmogoroff's formula (Brockwell and Davis, 1991, Equation (5.8.1)). In Theorems 7, 8, and 9 this result will be established for a more general family of processes. Since $g_Z(n)$ is the generalized variance of the process Z_t/σ_A, it will be referred to as the standardized generalized variance. Without loss of generality we will let $\sigma_A = 1$.

THEOREM 2. *Let* Z_t *denote fractionally differenced white noise with parameter* $d \in (-\frac{1}{2}, \frac{1}{2})$ *and* $d \neq 0$. *Then* $g_Z(n) \to \infty$.

PROOF. As in McLeod (1978), $g_Z(n) = \prod_{k=0}^{n-1} \sigma_k^2$, where σ_k^2 denotes the variance of the error in the linear predictor of Z_{k+1} using $Z_k, \ldots, Z_1$. From the Durbin–Levinson recursion,

$$\sigma_k^2 = \begin{cases} \gamma_Z(0) & k = 0 \\ \sigma_{k-1}^2(1 - \phi_{k,k}^2) & k > 0 \end{cases}$$

where $\phi_{k,k}$ denotes the partial autocorrelation function at lag k. For the special case $p = q = 0$ in (4), Hosking (1981) showed that $\phi_{k,k} = d/(k-d)$ and $\gamma_Z(0) = (-2d)!/(-d)!^2$. Using the Durbin–Levinson recursion,

$$\sigma_k^2 = \frac{k!(k-2d)!}{(k-d)!^2}.$$

Applying the Stirling approximation to $\log(t!)$ for large t, $\log(t!) \sim (t + \frac{1}{2})\log(t) - t + \frac{1}{2}\log(2\pi)$, yields $\log(\sigma_k^2) \sim a(k)$, where

$$a(k) = \left(k + \frac{1}{2}\right) \log\left\{\frac{k(k-2d)}{(k-d)^2}\right\} + 2d \log\left(\frac{k-d}{k-2d}\right).$$

Since σ_k^2 is a monotone decreasing sequence and, for $d \neq 0$, $\sigma_k^2 > 1$, it follows that $\log(\sigma_k^2)$ is a positive monotone decreasing sequence. By Stirling's approximation $\log(\sigma_k^2)a(k) \to 1$ as $k \to \infty$. So for large k, $a(k)$ must be a monotone decreasing sequence of positive terms. Expanding $a(k)$ and simplifying

$$a(k) = \left(k + \frac{1}{2}\right) \log\left(1 - \frac{2d}{k}\right) + 2\left(k + \frac{1}{2}\right) \log\left\{1 + \frac{d}{k} + \left(\frac{d}{k}\right)^2 + \ldots\right\}$$

$$+ 2d \log\left(1 + \frac{d}{k - 2d}\right) = \frac{d^2}{k} + \mathrm{O}\left(\frac{1}{k^2}\right)$$

where the expansion $\log(1 + x) = x + x^2/2 + x^3/3 + \ldots$, $|x| < 1$, has been used. Hence,

$$ka(k) \to d^2 \qquad \text{as } k \to \infty \tag{5}$$

and by the theorem given by Knopp (1951, Section 80, p. 124), $\sum a(k)$ diverges for $d \neq 0$. So for $d \neq 0$, $\sum \log(\sigma_k^2)$ diverges and consequently so does $g_Z(n)$. ■

Equation (5) shows that $\sigma_k^2 = 1 + \mathrm{O}(k^{-1})$ which implies that σ_k^2 decays very slowly. The divergence of $g_Z(n)$ can be slow. See Table I.

2. HYPERBOLIC DECAY TIME SERIES

The stationary, purely non-deterministic time series Z_t is said to be a hyperbolic decay time series with decay parameter α, $\alpha \in (0, 2)$, $\alpha \neq 1$, if for large k

$$\gamma_Z(k) \sim c_\gamma k^{-\alpha} \tag{6}$$

where $c_\gamma > 0$ for $\alpha \in (0, 1)$ and $c_\gamma < 0$ for $\alpha \in (1, 2)$. When $\alpha \in (1, 2)$ the time series is said to be antipersistent. As shown in the next theorem, antipersistent time series have a spectral density function which decays rapidly to zero near the origin. The term antipersistent was coined by Mandelbrot (1983) for FGN processes with Hurst parameter $0 < H < 1/2$. Hyperbolic decay time series include both FGN time series with parameter $H = 1 - \alpha/2$, $H \in (0, 1)$, $H \neq 1/2$, and FARMA time series with parameter $d = 1/2 - \alpha/2$, $d \in (-1/2, 1/2)$, $d \neq 0$.

THEOREM 3. *The spectral density function of hyperbolic decay time series satisfies (2).*

TABLE I

GENERALIZED VARIANCE $g_Z(n)$ FOR $n = 10^k$, $k = 0, 1, \ldots, 7$, OF FRACTIONALLY DIFFERENCED WHITE NOISE Z_t WITH PARAMETER d

d	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
−0.4	1.1831	1.6225	2.3318	3.3685	4.8688	7.0375	10.1725	14.7059
−0.1	1.0145	1.0366	1.0607	1.0854	1.1107	1.1365	1.1630	1.1901
0.1	1.0195	1.0434	1.0678	1.0927	1.1181	1.1442	1.1708	1.1990
0.4	2.0701	3.1588	4.5923	6.6417	9.6009	13.8775	20.0591	28.9951

PROOF. Beran (1994) established this result when $\alpha \in (0, 1)$ as was noted above in Equation (2). However, the theorem of Zygmund (1968, Section V.2) used by Beran (1994, Theorem 2.1) does not apply to the case where $\alpha \in (1, 2)$.

Let Y_t have the spectral density $f_Y(\lambda) = c_f \lambda^{\alpha-1}$, $\alpha \in (1, 2)$.

$$\gamma_Y(k) = 2\int_0^{\pi} c_f \lambda^{\alpha-1} \cos(\lambda k)\, d\lambda$$

$$= 2c_f k^{-\alpha} \int_0^{k\pi} u^{\alpha-1} \cos(u)\, du$$

Using Mathematica,

$$\int_0^{\infty} u^{\alpha-1} \cos(u)\, du = \frac{\sqrt{\pi}\Gamma(\alpha/2)}{(1/4)^{(\alpha-1)/2}\Gamma\{(1-\alpha)/2\}}$$

and so $\gamma_Y(k) \to c_{/\gamma} k^{-\alpha}$, where $c_\gamma = 2c_f \sqrt{\pi}\Gamma(\alpha/2)/[(1/4)^{(\alpha-1)/2}\Gamma\{(1-\alpha)/2\}] < 0$.

Assume $f_Z(\lambda)$ satisfies Equation (2) and we will derive (6). Since $f_Z(\lambda)/(c_f\lambda^{\alpha-1}) \to 1$ as $\lambda \to 0$ there exists λ_0 such that, for all $\lambda < \lambda_0$, $c_f \lambda^{\alpha-1} < 1$ and $|f_Z(\lambda)/(c_f\lambda^{\alpha-1}) - 1| < \epsilon/(2\pi)$. Hence, for all $\lambda < \lambda_0$, $|f_Z(\lambda) - f_Y(\lambda)| < \epsilon/(2\pi)$. Consider the systematically sampled series $Z_{t,l} = Z_{tl}$ for $l \geqslant 1$. Then $Z_{t,l}$ has spectral density function $f_Z(\lambda/l)$. Let $L = \pi/\lambda_0$. Then $|f_Z(\lambda/l) - f_Y(\lambda)| < \epsilon/(2\pi)$ for $\lambda \in (0, \pi)$ provided that $l > L$. Hence, for any $l > L$,

$$|\gamma_Z(kl) - \gamma_Y(k)| < 2\int_0^{\pi} |\cos(\lambda k)| |f_Z\left(\frac{\lambda}{l}\right) - f_Y(\lambda)| d\lambda$$

$$< 2\int_0^{\pi} \left| f_Z\left(\frac{\lambda}{l}\right) - f_Y(\lambda)\right| d\lambda$$

$$< \epsilon.$$

This shows that (2) implies (6). Since the spectral density uniquely defines the autocovariance function, the theorem follows. ∎

Hyperbolic decay time series are self-similar: aggregated series are hyperbolic with the same parameter as the original.

THEOREM 4. *Let Z_t satisfy Equation (6). Then so does Y_t, where $Y_t = \sum_{j=1}^{m} Z_{(t-1)m+j}/m$ and m is any value.*

PROOF. For large l,

$$\begin{aligned}\gamma_Y(l) &= m^{-2}\operatorname{cov}\left(\sum_{h=1}^{m} Z_{(t-1)m+h}, \sum_{k=1}^{m} Z_{(t-1+l)m+k}\right)\\ &\sim m^{-2}\sum_{h=1}^{m}\sum_{k=1}^{m} c_\gamma(k+ml-h)^{-\alpha}\\ &\sim m^{-2}\sum_{h=1}^{m}\sum_{k=1}^{m} c'_\gamma l^{-\alpha}\left(1+\frac{k-h}{ml}\right)^{-\alpha}\\ &\sim c'_\gamma l^{-\alpha}\end{aligned}$$

where $c'_\gamma = m^{-\alpha}c_\gamma$. ■

3. DUALITY

Duality has provided insights into linear time series models (Finch, 1960; Pierce, 1970; Cleveland, 1972; Box and Jenkins, 1976; Shaman, 1976; McLeod, 1977, 1984). In general, the dual of the stationary invertible linear process $Z_t = \psi(B)A_t$ is defined to be $\psi(B)\ddot{Z}_t = A_t$, where $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$ and B is the backshift operator on t. Equivalently, if Z_t has spectral density $f_Z(\lambda)$ then the dual has spectral density proportional to $1/f_Z(\lambda)$ with the constant of proportionality determined by the innovation variance. Thus in the case of a FARMA (p, q) with parameter d the dual is a FARMA (q, p) with parameter $-d$. The next theorem generalizes this to the hyperbolic case.

THEOREM 5. *The dual of a hyperbolic decay time series with decay parameter α is another hyperbolic decay series with decay parameter $2-\alpha$.*

PROOF. The spectral density near zero of the dual of a hyperbolic decay time series with parameter α is $1/(c_f\lambda^{\alpha-1}) = c_f^{-1}\lambda^{(2-\alpha)-1}$ which implies a hyperbolic process with parameter $2-\alpha$. ■

THEOREM 6. *The time series Z_t satisfying (6) may be written in random shock form as $Z_t = A_t + \sum\psi_l A_{t-l}$ where $\psi_l \sim c_\psi l^{-(1+\alpha)/2}$ and $c_\psi > 0$ for $\alpha \in (0, 1)$ and $c_\psi < 0$ for $\alpha \in (1, 2)$ and in inverted form as $Z_t = A_t + \sum\pi_l Z_{t-l}$ where $\pi_l \sim c_\pi l^{-(3-\alpha)/2}$ and $c_\pi > 0$ for $\alpha \in (0, 1)$ and $c_\pi < 0$ for $\alpha \in (1, 2)$.*

PROOF. The case $\alpha \in (0,\ 1)$ was established in Theorem 1. When $\alpha \in (1,\ 2)$ the random shock coefficients are given by

$$\psi_l \sim -c_{2-\pi} l^{-\{3-(2-\alpha)\}/2}$$

$$\sim c_\psi l^{-(1+\alpha)/2}$$

where $c_\psi = c_{2-\pi}$. Similarly for the inverted form. ■

4. GENERALIZED VARIANCE

For ARMA process Z_t, $\lim g_Z(n)$ is finite and has been evaluated by Finch (1960) and McLeod (1977). McLeod (1977, Equation (2) showed $g_Z(n) = m_Z + O(r^n)$, where $r \in (0, 1)$. The evaluation of this limit uses the theorem of Grenander and Szegö (1984, Section 5.5) which only applies to the case where the spectral density $f_Z(\lambda)$, $\lambda \in [0, 2\pi)$, satisfies the Lipschitz condition $|f'_Z(\lambda_1) - f'_Z(\lambda_2)| < K|\lambda_1 - \lambda_2|^\varsigma$, for some $K > 0$ and $0 < \zeta < 1$. Since, when $\alpha \in (0, 1)$, $f'_Z(\lambda)$ is unbounded, this condition is not satisfied.

LEMMA 1. *Let X_t and Y_t be any independent stationary processes with positive innovation variance and let $Z_t = X_t + Y_t$. Then $G_Z(n) > G_X(n)$.*

PROOF. This follows directly from the fact that the one-step predictor error variance of Z_t cannot be less than that of X_t. ■

THEOREM 7. *Let Z_t denote a strongly persistent time process defined in Equation (2). Then $g_Z(n) \to \infty$.*

PROOF. Since $Z_t = \sum \psi_k A_{t-k}$, where A_t is white noise unit variance, we can find a q such that the process Y_t where

$$Y_t = \sum_{k=q+1}^{\infty} \psi_k A_{t-k}$$

has all autocovariances non-negative and satisfying Equation (1). By using the comparison test for a harmonic series, it must be possible to find an N such that, for $n > N$, the covariance matrix $\Gamma_Y(n)$ has every row-sum greater than Ξ, for any $\Xi > 0$. It then follows from the Frobenius theorem (Minc and Marcus, 1964, p. 152) that the largest eigenvalue of $\Gamma_Y(n)$ tends to ∞ as $n \to \infty$. Assume now that $\inf f_Y(\lambda) = m$ where $m > 0$ and let m_n denote the smallest eigenvalue of $\Gamma_Y(n)$ and ζ_n denote the corresponding eigenvector. Then

$$
\begin{aligned}
m_n &= m_n \zeta_n' \zeta_n \\
&= \zeta_n' \Gamma_Y(n) \zeta_n \\
&= \int_{-\pi}^{\pi} \sum_h \sum_l \zeta_{n,h} \zeta_{n,l}\, e^{-i\lambda(h-l)} f(\lambda)\, d\lambda \\
&\geqslant 2\pi m.
\end{aligned}
$$

So $m_n \geqslant 2\pi m$ hence $g_Y(n) \to \infty$ as $n \to \infty$. By Lemma 1, $g_Z(n) \to \infty$ also.

For the more general case where $m = 0$, consider a process with spectral density function $f(\lambda) + \epsilon$, where $\epsilon > 0$. Let $g_\epsilon(n)$ denote the standardized covariance determinant of n successive observations of this process. So $g_\epsilon(n) \to \infty$ as $n \to \infty$ for every $\epsilon > 0$. The autocovariance function corresponding to $f(\lambda) + \epsilon$ is

$$
\gamma_\epsilon(k) = \begin{cases} \gamma_Z(0) + 2\pi\epsilon & k = 0 \\ \gamma_Z(k) & k \neq 0. \end{cases}
$$

By continuity of the autocovariance function with respect to ϵ, $\lim g_\epsilon(n) \to g_Z(n)$ as $\epsilon \to 0$. Let $\Xi > 0$ be chosen as large as we please and let $\delta > 0$. Then for any $\epsilon > 0$ there exists an $N(\epsilon)$ such that, for all $n \geqslant N(\epsilon)$, $g_\epsilon(n) > \Xi + \delta$. By continuity, there exists an ϵ_0 such that $g_Z\{N(\epsilon_0)\} > g_{\epsilon 0}\{N(\epsilon_0)\} - \delta$. Hence $g_Z\{N(\epsilon_0)\} > \Xi$. Since $g_Z(n+1) = g_Z(n)\sigma_n^2$, where $\sigma_n^2 > 1$ is the variance of the error of the linear predictor of Z_{n+1} given $Z_n, \ldots, Z_1$, we see that $g_Z(n)$ is non-decreasing. It follows that $g_Z(n) > \Xi$ for all $n > N(\epsilon_0)$.

Using a theorem of Grenander and Szegö (1984) this result is easily generalized to any stationary time series Z_t for which $\sum \gamma_Z(k) = \infty$.

THEOREM 8. *Let Z_t denote a time series for which $f_Z(\lambda) \to \infty$ as $\lambda \to 0$. Then $g_Z(n) \to \infty$.*

PROOF. From Equation (10) of Grenander and Szegö (1984, Section 5.2), as $n \to \infty$, the largest eigenvalue of $\sigma_a^{-2}\Gamma_Z(n)$ approaches $\sup f_Z(\lambda) = \infty$ while the smallest eigenvalue approaches $2\pi m$, where $m = \inf f(\lambda)$. Note that Grenander and Szegö's Equation (10), Section 5.2, applies directly to unbounded spectral densities as is pointed out by Grenander and Szegö in the sentence immediately following Equation (10), Section 5.2. If it is assumed that $m > 0$, then the largest eigenvalue tends to infinity and the smallest one is bounded by $2\pi m$ as $n \to \infty$. Hence, $g_Z(n) \to \infty$ for this special case. The more general case where $m = 0$ is handled as in Theorem 7. ■

In the case of ARMA models, the asymptotic covariance determinant of the dual and primal are equal (Finch, 1960). Since the hyperbolic decay time series are approximated by high order AR and MA models, it might be expected that this property holds for hyperbolic series too. Theorem 9 which uses Lemma 2 proves that this is the case.

LEMMA 2. *Let* $X_t = A_t + \sum_1^\infty \psi_l A_{t-l}$. *Let* $X_t(q) = A_t + \sum_1^q \psi_l A_{t-l}$, *and let* $g_q(n)$ *denote its standardized covariance determinant. Then, for any* $l > 0$, $g_{q+l}(n) \geqslant g_q(n)$.

PROOF. This follows directly from the fact that the one-step predictor error variance of $X_t(q+l)$ cannot be less than that of $X_t(q)$. ■

THEOREM 9. *For hyperbolic decay antipersistent time series* Z_t, $g_Z(n) \to \infty$.

PROOF. Since the dual of the antipersistent time series Z_t with parameter $2-\alpha$, $\alpha \in (0, 1)$, is a strongly persistent time series $\ddot{Z}_t$ with parameter α, $\ddot{Z}_t$ may be represented in inverted form $\ddot{Z}_t = A_t + \sum \pi_k \ddot{Z}_{t-k}$, where A_t is white noise and, for large k, $\pi_k \sim c_\pi k^{-(3-\alpha)/2}$. So the antipersistent time series Z_t can be written $Z_t = A_t - \sum \pi_k A_{t-k}$. Let $\ddot{g}_L(n)$ and $g_L(n)$ denote the covariance determinant of n successive observations in the AR(L) and MA(L) approximation to $\ddot{Z}_t$ and Z_t:

$$\ddot{Z}_t(L) = A_t + \sum_{k=1}^{L} \pi_k \ddot{Z}_{t-k}(L)$$

and

$$Z_t(L) = A_t - \sum_{k=1}^{L} \pi_k A_{t-k}.$$

By Theorem 7, for any $\Xi > 0$ and $\delta > 0$ there exists an N_1 such that, for $n > N_1$, $g_{\ddot{Z}}(n) > \Xi + \delta$. Since $\ddot{g}_k(n) \to \ddot{g}_Z(n)$ as $k \to \infty$ there exists a $K_1(n)$ such that $\ddot{g}_k(n) > \ddot{g}_Z(n) - \delta > \Xi$ for $k > K_1(n)$. From McLeod (1977), $\ddot{g}_k(n) = \ddot{g}_k(k)$ for $n \geqslant k$. Hence, for any $n > N_1$, $\ddot{g}_k(m) > \ddot{g}_Z(n) - \delta > \Xi$ for $k > K_1(n)$ and $m \geqslant k$. So $\ddot{g}_k(m) \to \infty$ as $k \to \infty$ and $m \geqslant k$.

Hence there exists K_2 such that $\ddot{g}_k(n) > \Xi + \delta$ for $k > K_2$ and $n \geqslant k$. For any k, $g_k(n) = \ddot{g}_k(n) + \mathrm{O}(r^n)$, where $0 < r < 1$ (McLeod, 1977). Let $k > K_2$. Then there exists an $N_2(k)$ such that, for all $n > N_2(k)$, $g_k(n) > \ddot{g}_k(n) - \delta > \Xi$. So $g_k(n) \to \infty$ as $k \to \infty$ and $n \geqslant k$.

For any n, $g_k(n) \to g_Z(n)$ as $k \to \infty$. So for any n there exists a $K_3(n)$ such that $g_Z(n) > g_k(n) - \delta$ for all $k > K_3(n)$. We have already established that there exists a K_4 such that $g_k(n) > \Xi + \delta$ for $k > K_4$ and $n \geqslant k$. Holding n fixed for the moment, let $h > k$. By Lemma 2, $g_h(n) \geqslant g_k(n)$. By continuity, since $h > K_4$, $g_Z(n) > g_h(n) - \delta$. Since $g_h(n) > \Xi + \delta$ it follows that $g_Z(n) > \Xi$. This establishes that $g_Z(n) \to \infty$ as $n \to \infty$. ■

5. CONCLUDING REMARKS

Theorems 7 and 9 show that hyperbolic decay time series, even antipersistent ones, exhibit a type of long-range dependence. The asymptotic standardized generalized variance is infinite. This implies that the variance of the one-step linear predictor based on the last k observations decays very slowly compared with the ARMA case where the decay to the innovation variance occurs exponentially fast. Theorem 8 shows that this is a more general notion of long-range dependence than the customary one.

Yakowitz and Heyde (1997) show that non-linear Markov processes can also exhibit strongly persistent hyperbolic decay in the autocorrelation function. Hence a better term for long-memory time series might be strongly persistent hyperbolic decay series. It is then clear that the long-range dependent aspect is merely a characterization of the autocorrelation structure.

REFERENCES

BERAN, J. (1994) *Statistics for Long Memory Processes*. London: Chapman and Hall.

—— (1995) Maximum likelihood estimation of the differencing parameter for invertible short and long memory autoregressive integrated moving average models. *J. R. Stat. Soc. B* 57, 659–72.

BOX, G. E. P. and JENKINS, G. M. (1976) *Time Series Analysis: Forecasting and Control*, 2nd Edn. San Francisco, CA: Holden-Day.

BROCKWELL, P. J. and DAVIS, R. A. (1991) *Time Series: Theory and Methods*, 2nd Edn. New York: Springer.

CLEVELAND, W. S. (1972) The inverse autocorrelations of a time series and their applications. *Technometrics* 14, 227–93.

DIEBOLD, F. X. and RUDEBUSCH, G. D. (1989) Long memory and persistence in aggregate output. *J. Monet. Econ.* 24, 189–209.

FINCH, P. D. (1960) On the covariance determinants of autoregressive and moving average models. *Biometrika* 47, 194–96.

GRAHAM, R. L., KNUTH, D. E. and PATASHNIK, O. (1989) *Concrete Mathematics*. Reading, MA: Addison-Wesley.

GRANGER, C. W. J. and JOYEUX, R. (1980) An introduction to long-range time series models and fractional differencing. *J. Time Ser. Anal.* 1, 15–30.

GRENANDER, U. and SZEGÖ, G. (1984) *Toeplitz Forms and Their Applications*, 2nd Edn. New York: Chelsea.

HIPEL, K. W. and MCLEOD, A. I. (1994) *Time Series Modelling of Water Resources and Environmental Systems*. Elsevier: Amsterdam.

HOSKING, J. R. M. (1981) Fractional differencing. *Biometrika* 68, 165–76.

KNOPP, K. (1951) *Theory and Application of Infinite Series*, 2nd Edn. New York: Hafner.

MANDELBROT, B. B. M. (1983) *The Fractal Geometry of Nature*. San Francisco, CA: Freeman.

MCLEOD, A. I. (1977) Improved Box–Jenkins estimators. *Biometrika* 64, 531–34.

—— (1984) Duality and other properties of multiplicative seasonal autoregressive-moving average models. *Biometrika* 71, 207–11.

MINC, H. and MARCUS, M. (1964) *A Survey of Matrix Theory and Matrix Inequalities*. Boston, MA: Prindle, Weber and Schmidt.

PIERCE, D. A. (1970) A duality between autoregressive and moving average processes concerning their parameter estimates. *Ann. Stat.* 41, 722–26.

SHAMAN, P. (1976) Approximations for stationary covariance matrices and their inverses with application to ARMA models. *Ann. Stat.* 4, 292–301.

YAKOWITZ, S. J. and HEYDE, C. C. (1997) Long-range dependency effects with implications for forecasting and queueing inference. Unpublished manuscript.

ZYGMUND, A. (1968) *Trigonometric Series*. London: Cambridge University Press.