STAT 2857 — Assignment 3

Problem 1. Let X be a continuous non-negative random variable with density function f, and let $Y := X^n$. Find f_Y, the probability density function of Y.

Problem 2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months)

$$f(x) = \begin{cases} cxe^{-x/2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

What is the probability that the system functions for at least 5 months?

Problem 3. Let X be a discrete random variable with probability function $\mathbb{P}\{X = x\} = \frac{2}{3^x}$ for $x = 1, 2, 3, \ldots$ What is the probability that X is even?

Problem 4. For a certain discrete random variable on the non-negative integers, the probability function satisfies the relationships $\mathbb{P}\{0\} = \mathbb{P}\{1\}$ and $\mathbb{P}\{k+1\} = \frac{1}{k}\mathbb{P}\{k\}$ for $k = 1, 2, 3, \ldots$ Find $\mathbb{P}\{0\}$.

Problem 5. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0 & \textit{otherwise}. \end{cases}$$

Calculate $\mathbb{P}\{|X - \frac{1}{2}| > \frac{1}{4}\}$.

Problem 6. Let X be a random variable with distribution function

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{8}, & 0 \leq x < 1 \\ \dfrac{1}{4} + \dfrac{x}{8}, & 1 \leq x < 2 \\ \dfrac{3}{4} + \dfrac{x}{12}, & 2 \leq x < 3 \\ 1, & x \geq 3. \end{cases}$$

Calculate $\mathbb{P}\{1 \leq X \leq 2\}$.

Problem 7. Let X have the density function $f(x) = 3x^2/\theta^3$ for $0 < x < \theta$, and $f(x) = 0$ otherwise. If $\mathbb{P}\{X > 1\} = 7/8$, find the value of θ.

Problem 8. X is a continuous random variable with density function $f(x) = ce^{-x}$, $x > 1$. Find $\mathbb{P}\{X < 3 | X > 2\}$.

Problem 9. A discrete integer-valued random variable has the following probability function: $\mathbb{P}\{X = n\} = a_n - a_{n+1}$, where the a_n's are numbers which satisfy the following conditions:

(i) $a_0 = 1$

(ii) $a_0 > a_1 > a_2 > \cdots > a_k > a_{k+1} > \cdots > 0$.

Find the probability $\mathbb{P}\{X \leq 5 | X > 1\}$.

Problem 10. a) Show that the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, for $x \in \mathbb{R}$, is a density function of a random variable X. This random variable is called *standard normal.*

b) Find the density function of $Y = \sigma X + \mu$ where μ and σ are constants with $\sigma > 0$.

c) Find the density function of $Y = e^X$. In this case, the random variable Y is called *lognormal.*

Problem 11. a) Show that the function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

where λ is a positive constant, is a density function of a random variable X. We say that X is *exponentially distributed with parameter* λ.

b) If X is an exponential random variable with parameter λ, and $c > 0$, show that cX is exponential with parameter λ / c.

c) Suppose X has an exponential distribution with parameter 1 and $Y = \log(X)$. Find the distribution function of X. This is the *double exponential distribution.*

Problem 12. A point is chosen at random (uniformly) on a line segment of length 1. The point divides the segment into two smaller segments. Find the probability that the ratio of the shorter to the longer segment is less than 1/4.

Hint. By definition, for a continuous r.v. X we have $P(a \leq X \leq b) = \int_a^b f(x)\,dx$, where $f(x)$ is the p.d.f. of X. From here one can show (take it for granted) that for a "nice" subset E of the real line one has $P(X \in E) = \int_E f(x)\,dx$.

Problem 13. Let X be an absolutely continuous random variable having cumulative distribution function F. Assume F is strictly increasing function. Define the random variable $Y := F(X)$. Show that Y is uniformly distributed over $[0, 1]$.

Problem 14. People enter a gambling casino at a rate of 1 for every 2 minutes.
(a) What is the probability that no one enters between 12 : 00 and 12 : 05?
(b) What is the probability that at least 4 people enter the casino during that time?

Problem 15. A typing agency employs 2 typists. The average number of errors per article is 3 when typed by the first typist and 4.2 when typed by the second. If your article is equally likely to be typed by either typist, approximate the probability that it will have no errors.

Problem 16. Suppose X has density function $x/2$ for $0 < x < 2$, and 0 otherwise. Find the density function of $Y = X(2 - X)$ by computing $P(Y \geq y)$ and then differentiating.

Problem 17. A weather channel has the local forecast on the hour and at 10, 25, 30, 45, and 55 minutes past. Suppose you wake up in the middle of the night and turn on the TV. Let X be the time you have to wait until the beginning of the next local forecast, measured in minutes. Find the density function of X.

Problem 18. (Fun problem) An airplane has 100 seats and is fully booked. Every passenger has an assigned seat. Passengers board one by one. Unfortunately, passenger 1 loses his boarding pass and can't remember his seat. So he picks a seat at random (among the unoccupied ones) and sits on it. Other passengers come aboard and if their seat is taken, they choose a vacant seat at random. What is the probability that the last passenger ends up on his own seat?