STAT 2857 — Assignment 4

Problem 1. a) Find the moments of the random variable X if its moment generating function is $M_X(t) = (1 - p_1 - p_2) + p_1 e^t + p_2 e^{2t}$.
b) What is the variance of X?

Problem 2. Find the probability $P(X \leq 1.23)$ if X has moment generating function $M_X(t) = (1 - p + pe^t)^5$.

Problem 3 Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{1}{30}x(1+3x), & 1 < x < 3 \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}\left\{\frac{1}{X}\right\}$.

Problem 4. If $f(x) = (k+1)x^2$ for $0 < x < 1$, find the moment generating function of X.

Problem 5. Find the probability $P(X > 1.5)$ if X has moment generating function $M_X(t) = \frac{1}{1-t/2}\left(\frac{1}{3} + \frac{2}{3}e^t\right)^4$. *Hint:* Use Theorem 124.

Problem 6. A game is played where a fair coin is tossed until the first tail occurs. The probability x tosses will be needed is $f(x) = 0.5^x$, $x = 1, 2, 3, \ldots$ You win $\$2^x$ if x tosses are needed for $x = 1, 2, 3, 4, 5$ but lose \$256 if $x > 5$. Determine your expected winnings.

Problem 7. According to an airline report, roughly 1 piece of luggage out of every 2000 that are checked is lost. Suppose that a frequent-flying businesswoman is checking 1200 bags over the course of the next year. What approximately is the probability that she will lose 2 or more pieces of luggage.

Problem 8. When a machine is improperly adjusted it has probability 0.15 of producing a defective item. Each day the machine is run until three defective items are produced. If this occurs, the machine is stopped and checked for adjustment. What is the probability that an improperly adjusted machine will produce five or more items before it is stopped? What is the average number of items an improperly adjusted machine will produce before being stopped?

Problem 9. Oak Hill has $74,806$ registered automobiles. A city ordinance requires each to display a bumper decal showing that the owner paid an annual wheel tax of \$50. By law, new decals need to be purchased during the month of the owner's birthday. This year's budget assumes that at least $\$306,000$ in decal revenue will be collected in November. What is the probability that the taxes reported in that month will be less than anticipated and produce a budget shortfall? (It is reasonable to assume that the probability that a birthday falls in November is $1/12$.)

Problem 10. Let Y be a random variable uniformly distributed in $[0, 5]$. Find the probability that the roots of the equation $4x^2 + 4xY + Y + 2 = 0$ are both real.

Problem 11. A life insurer has created a special one-year term insurance policy for a pair of business people who travel to high risk locations. The insurance policy pays nothing if neither die in the year, \$100,000 if exactly one of the two die, and $\$K > 0$ if both die. The insurer determines that there is a probability 0.1 that at least one of the two will die during the year and a probability of 0.08 that exactly one of the two will die during the year. You are told that the standard deviation of the payout is \$74,000. Find the expected payout for the year on this policy.

Let $q \in (0, 1)$. The q-th percentile of a random variable having distribution function F is the smallest value m such that $F(m) \geq q$.

Problem 12. Find the q-th percentile of
a) uniformly distributed random variable in $[a, b]$;
b) exponential random variable with parameter λ.
c) Geometric random variable with parameter p.
The 1/4-percentile is called the *first quartile.* The 1/2-percentile is called the *median* or the s*econd quartile.* The 3/4-percentile is called the *third quartile.*

Hint. For part c) use the functions *floor of* x: $\lfloor x \rfloor$ defined to be the largest integer not larger than x; and the *ceiling of* x: $\lceil x \rceil$ defined to be the smallest integer not smaller than x.

Problem 13. Let X be a normal random variable with mean 2 and variance 2.
a) Find the probabilities $P(1 \leq X \leq 2.5)$ and $P(X > 3)$.
b) Find the real number z_α such that $P(X \geq z_\alpha) = \alpha$ for $\alpha = 0.10, 0.05$.
c) Find the q-th percentile of X for $q = 0.25, 0.50, 0.75$.

Problem 14. An airline company sells 200 tickets for a plane with 198 seats, knowing that the probability a passenger will not show up for the flight is 0.01. Assume that the ticket holders act independently of each other.

a) Use the appropriate approximation to compute the probability they will have enough seats for all the passengers who show up.

b) How good is your approximation? (Use Theorem 136 from the course notes)

c) Find the exact probability

Problem 15. An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If (2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly k selections?

Problem 16. The life of a certain type of automobile tire is normally distributed with mean $34,000$ miles and standard deviation 4000 miles.

(a) What is the probability that such a tire lasts over $40,000$ miles?

(b) What is the probability that it lasts between $30,000$ and $35,000$ miles?

(c) Given that it has survived $30,000$ miles, what is the probability that it survives another $10,000$ miles?

Problem 17. If a sequence $A_1, A_2, A_3, \ldots$ of events in Ω is decreasing, show that the intersection of all events in the sequence has probability

$$P\Big(\bigcap_{k=1}^{\infty} A_k\Big) = \lim_{k\to\infty} P(A_k).$$

Problem 18. The following table uses data concerning the percentages of male and female full time workers whose annual salaries fall in different ranges. Suppose that random samples of 200 male and 200 female full time workers are chosen. Approximate the probability that

(a) at least 70 of the women earn $\$25,000$ or more;

(b) at most 60 percent of the men earn $\$25,000$ or more;

(c) at least three-fourths of the men and at least half the women earn $\$20,000$ or more.

Earnings range	Percentage of females	Percentage of males
≤9999	8.6	4.4
10,000–19,999	38.0	21.1
20,000–24,999	19.4	15.8
25,000–49,999	29.2	41.5
≥50,000	4.8	17.2

Problem 19. You are tied to your chair. I have an empty gun, a revolver. I put two bullets into the barrel. I close the barrel and spin it. I put the gun to your head and pull the trigger. Click. You got lucky! Now I am going to pull the trigger one more time. Would you prefer that I spin the barrel again or that I just pull the trigger if

a) I put the bullets into two adjacent chambers?

b) I put the bullets not next to each other?