Statistical Sciences 437b **January 13, 2003**

Assignment 1: The M/G/1 Queueing System
Due: January 27

1. Verify Equation $\pi_j = \pi_0 a_j + \sum_{i=1}^{j+1} \pi_i a_{j-i+1} \ (j = 0,1,2,...)$ for $j = 0,1,2,3$.

2. For the M/G/1 queue, let X_n denote the number in the system left behind by the nth departure.

(a) If

$$X_{n+1} = \begin{cases} X_n - 1 + A, X_n \geq 1 \\ A, X_n = 0 \end{cases}$$

what does A represent ?

(b) Rewrite the preceding as

$$X_{n+1} = X_n - \delta_n + A \qquad (1)$$

where

$$\delta_n = \begin{cases} 1, (X_n > 0), \\ 0, (X_n = 0). \end{cases}$$

Take expectations and let $n \to \infty$ in Equation (1) to obtain

$$E[\delta_n] = E[a] = \lambda E[S] = \frac{\lambda}{\mu} = \rho$$

© Square both sides of Equation (1), take expectations, and then let $n \to \infty$ to obtain ($E[X_{n+1}] = E[X_n] = L$)

$$L = \rho + \frac{\rho^2 + \lambda^2 \sigma_S^2}{2(1-\rho)}.$$

3. From equation $\pi(z) = \frac{(1-\rho)(1-z)a(z)}{a(z)-z}$ derive the P-K formula $L = \rho + \frac{\rho^2 + \lambda^2 \sigma_S^2}{2(1-\rho)}$ by using that fact that $L = \pi'(1)$.

Hint: Use L'Hopital's rule twice, and $a''(1) = \lambda^2 \sigma_s^2 + \rho^2$

$$(a''(1) = \sum_{i=1}^{\infty} i(i-1)a_i = \sum_{i=1}^{\infty} i^2 a_i - \sum_{i=1}^{\infty} i a_i = Var(a) + (E[a])^2 - E[a]$$
$$= \rho + \lambda^2 \sigma_s^2 + \rho^2 - \rho = \lambda^2 \sigma_s^2 + \rho^2)$$

4. In a small post office there is a single person serving. Seventy percent of customers take 1 min to serve, 20 percent take 3 min and 10 percent take 10 min. Calculate the average time spent in the post office, und the average number of people in the post office, when customers arrive at an average rate of (a) one per 3 min, (b) one per 4 min, (c) one per 5 min.

5. Using software package ZEDNET compare the mean response time and the mean queue length of the M/D/1, the M/M/1, and the M/G/1 system ($C_s = 0.5$ and $C_s = 2.5$) with $\lambda = 0.8\ \text{sec}^{-1}$ and $\mu = 1\ \text{sec}^{-1}$. Give a comprehensible explanation why the values are different although the arrival rate λ and service rate μ are always the same.