CHAPTER 3

3.1

(a) $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix}$; $A' = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

(b) $A'A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 8 & 16 \\ 8 & 5 & 8 \\ 16 & 8 & 21 \end{bmatrix}$

(c) tr(A) = 2 + 2 + 4 = 8 ; $\text{tr}(A'A) = 17 + 5 + 21 = 43$

(d) det(A) = (2)(2)(4) + (3)(1)(1) + (0)(2)(2) – (2)(2)(1) – (1)(2)(2) – (3)(0)(4) = 11

$\det(A'A)$ = (17)(5)(21)+(8)(8)(16)+(8)(8)(16)–(16)(5)(16)–(8)(8)(17)–(8)(8)(21)
= 121

3.2

(a) $X'X = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $(X'X)^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$; $X'\boldsymbol{y} = \begin{bmatrix} 19 \\ 1 \\ 5 \end{bmatrix}$;

$(X'X)^{-1}X'\boldsymbol{y} = \begin{bmatrix} 4.75 \\ 0.25 \\ 1.25 \end{bmatrix}$

(b) Diagonal matrices; the diagonal elements are the same.

3.3

(a) $$X'X = \begin{bmatrix} 5 & \sum_{i=1}^{5} x_{i1} & \sum_{i=1}^{5} x_{i2} \\ \sum_{i=1}^{5} x_{i1} & \sum_{i=1}^{5} (x_{i1})^2 & \sum_{i=1}^{5} x_{i1}x_{i2} \\ \sum_{i=1}^{5} x_{i2} & \sum_{i=1}^{5} x_{i1}x_{i2} & \sum_{i=1}^{5} (x_{i2})^2 \end{bmatrix}$$

(b) These quantities are entries in the $X'X$ matrix; see (a).

3.4

(a) det(A) = (2)(2) - (-1)(-1) = 5

(b) The eigenvalues are the solutions of the equation
$|A-\lambda I| = |(2-\lambda)(2-\lambda)-(-1)^2| = \lambda^2 - 4\lambda + 3 = 0$; they are 3 and 1.
The eigenvector corresponding to the eigenvalue $\lambda = 3$ is the solution to

$$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, $x_2 = -x_1$. Normalizing the length of the eigenvector to 1 leads to $2(x_1)^2 = 1$ and $x_1 = 1/\sqrt{2}$. Hence $x_2 = -1/\sqrt{2}$, and the eigenvector corresponding to the first eigenvector is given by $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

Similarly, the eigenvector corresponding to the second eigenvector is given by
$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

The matrix of eigenvectors is $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

(c) The spectral representation of the matrix A is given by

$$A = P\Lambda P' = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

(d) The eigenvalues of A are positive, hence the matrix A is positive definite. The matrix A can be a covariance matrix. The correlation matrix is given by

$$\begin{bmatrix} 2/2 & -1/\sqrt{(2)(2)} \\ -1/\sqrt{(2)(2)} & 2/2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix}.$$

3.5

(a) det(A) = (3)(4)(2) + (1)(1)(2) + (1)(1)(2) – (1)(1)(4) –(1)(1)(2) –(2)(2)(3) = 10.

The inverse is given by $A^{-1} = \begin{bmatrix} 0.4 & 0 & -0.2 \\ 0 & 0.5 & -0.5 \\ -0.2 & -0.5 & 1.1 \end{bmatrix}$. Check that $AA^{-1} = A^{-1}A = I$.

You can use a computer program to determine the inverse and also to check your calculations.

(b) The three eigenvalues are the solutions to the cubic equation $|A - \lambda I| = 0$. They are given by 5.8951, 2.3973, and 0.7076. The corresponding eigenvectors are the columns of the matrix

$$P = \begin{bmatrix} -0.4317 & 0.8857 & 0.1706 \\ -0.7526 & -0.4579 & 0.4732 \\ -0.4973 & -0.0759 & -0.8643 \end{bmatrix}$$

(c) The spectral representation of the matrix A is given by

$$A = P\Lambda P' = \begin{bmatrix} -0.4317 & 0.8857 & 0.1706 \\ -0.7526 & -0.4579 & 0.4732 \\ -0.4973 & -0.0759 & -0.8643 \end{bmatrix} \begin{bmatrix} 5.8951 & 0 & 0 \\ 0 & 2.3973 & 0 \\ 0 & 0 & 0.7076 \end{bmatrix} \begin{bmatrix} -0.4317 & 0.8857 & 0.1706 \\ -0.7526 & -0.4579 & 0.4732 \\ -0.4973 & -0.0759 & -0.8643 \end{bmatrix}'$$

(d) The eigenvalues of A are positive, hence the matrix A is positive definite. The matrix A can be a covariance matrix. The correlation matrix is given by

$$\begin{bmatrix} 3/3 & 1/\sqrt{(3)(4)} & 1/\sqrt{(3)(2)} \\ 1/\sqrt{(3)(4)} & 4/4 & 2/\sqrt{(4)(2)} \\ 1/\sqrt{(3)(2)} & 2/\sqrt{(4)(2)} & 2/2 \end{bmatrix} = \begin{bmatrix} 1 & 0.289 & 0.408 \\ 0.289 & 1 & 0.707 \\ 0.408 & 0.707 & 1 \end{bmatrix}.$$

3.6

(a) det(A) = (2)(4)(1) + (1)(0)(1) + (1)(0)(1) – (1)(4)(1) –(0)(0)(2) –(1)(1)(1) = 3.

The inverse is given by $A^{-1} = \begin{bmatrix} 4/3 & -1/3 & -4/3 \\ -1/3 & 1/3 & 1/3 \\ -4/3 & 1/3 & 7/3 \end{bmatrix}$. Check that $AA^{-1} = A^{-1}A = I$.

You can also use a computer program to check the calculations.

(b) The three eigenvalues are the solutions to the cubic equation $|A - \lambda I| = 0$. They are given by 4.4605, 2.2391, and 0.3004. The corresponding eigenvectors are the columns of the matrix

$$P = \begin{bmatrix} -0.4153 & -0.7118 & -0.5665 \\ -0.9018 & 0.4042 & 0.1531 \\ -0.1200 & -0.5744 & 0.8097 \end{bmatrix}$$

The spectral representation of the matrix A is given by

$$A = P\Lambda P' == \begin{bmatrix} -0.4153 & -0.7118 & -0.5665 \\ -0.9018 & 0.4042 & 0.1531 \\ -0.1200 & -0.5744 & 0.8097 \end{bmatrix} \begin{bmatrix} 4.4605 & 0 & 0 \\ 0 & 2.2391 & 0 \\ 0 & 0 & 0.3004 \end{bmatrix} \begin{bmatrix} -0.4153 & -0.7118 & -0.5665 \\ -0.9018 & 0.4042 & 0.1531 \\ -0.1200 & -0.5744 & 0.8097 \end{bmatrix}'$$

(c) The eigenvalues of A are positive, hence the matrix A is positive definite. The matrix A can be a covariance matrix. The correlation matrix is given by

$$\begin{bmatrix} 2/2 & 1/\sqrt{(2)(4)} & 1/\sqrt{(2)(1)} \\ 1/\sqrt{(2)(4)} & 4/4 & 0/\sqrt{(4)(1)} \\ 1/\sqrt{(2)(1)} & 0/\sqrt{(4)(1)} & 1/1 \end{bmatrix} = \begin{bmatrix} 1 & 0.354 & 0.707 \\ 0.354 & 1 & 0 \\ 0.707 & 0 & 1 \end{bmatrix}.$$

3.7

(a) det(A) = (2)(2)(6) + (1)(3)(3) + (1)(3)(3) – (3)(2)(3) –(1)(1)(6) –(3)(3)(2) = 0.

(b) The three eigenvalues are the solutions to the cubic equation $|A - \lambda I| = 0$. They are given by 9, 1, and 0. The corresponding eigenvectors are the columns of the matrix

$$P = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}$$

(c) The eigenvalues are nonnegative; hence the matrix A is semi-positive definite. The matrix A can be a covariance matrix. The correlation matrix is given by

$$\begin{bmatrix} 2/2 & 1/\sqrt{(2)(2)} & 3/\sqrt{(2)(6)} \\ 1/\sqrt{(2)(2)} & 2/2 & 3/\sqrt{(2)(6)} \\ 3/\sqrt{(2)(6)} & 3/\sqrt{(2)(6)} & 6/6 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.866 \\ 0.5 & 1 & 0.866 \\ 0.866 & 0.866 & 1 \end{bmatrix}$$

One eigenvalue is zero; hence there is a deterministic relationship among the three variables. The eigenvector corresponding to the eigenvalue 0 indicates the deterministic relationship. The linear combination $-y_1 - y_2 + y_3$ has variance zero.

3.8

(a) $AB = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 17 \\ 18 & 13 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 4 & 1 \\ 2 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 17 & 10 \\ 8 & 10 & 8 \\ 14 & 12 & 12 \end{bmatrix}$

3.9 An orthogonal matrix satisfies $PP' = P'P = I$. The eigenvectors of any symmetric matrix form an orthogonal matrix. Consider the matrix A in Exercise 3.6, for example. The eigenvectors are the columns in the matrix

$$P = \begin{bmatrix} -0.4153 & -0.7118 & -0.5665 \\ -0.9018 & 0.4042 & 0.1531 \\ -0.1200 & -0.5744 & 0.8097 \end{bmatrix}.$$

Then $PP' = P'P = I$.

3.10

$$X = \begin{bmatrix} 1 & 30 \\ 1 & 30 \\ 1 & 30 \\ 1 & 30 \\ 1 & 40 \\ 1 & 40 \\ 1 & 40 \\ 1 & 50 \\ 1 & 50 \\ 1 & 50 \\ 1 & 60 \\ 1 & 60 \\ 1 & 60 \\ 1 & 60 \end{bmatrix}; \quad \boldsymbol{y} = \begin{bmatrix} 55.8 \\ 59.1 \\ 54.8 \\ 54.6 \\ 43.1 \\ 42.2 \\ 45.2 \\ 31.6 \\ 30.9 \\ 30.8 \\ 17.5 \\ 20.5 \\ 17.2 \\ 16.9 \end{bmatrix}; \quad X'X = \begin{bmatrix} 14 & 630 \\ 630 & 30{,}300 \end{bmatrix};$$

$$(X'X)^{-1} = \begin{bmatrix} 1.10989 & -0.02308 \\ -0.02308 & 0.00051 \end{bmatrix} \qquad X'\boldsymbol{y} = \begin{bmatrix} 520.2 \\ 20{,}940.0 \end{bmatrix} \text{ and}$$

$$\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\boldsymbol{y} = \begin{bmatrix} 1.10989 & -0.02308 \\ -0.02308 & 0.00051 \end{bmatrix}\begin{bmatrix} 520.2 \\ 20{,}940.0 \end{bmatrix} = \begin{bmatrix} 94.1341 \\ -1.2662 \end{bmatrix}$$

3.11

(a) The distribution of $(y_1, y_2)'$ is bivariate normal with mean vector $(2,6)'$ and covariance matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) The conditional distribution of $(y_1, y_2)'$, given that $y_3 = 5$, is bivariate normal with mean vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix} + (1/3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}(y_3 - 4) = \begin{bmatrix} (2/3) + (1/3)y_3 \\ (22/3) - (1/3)y_3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 17/3 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - (1/3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 5/3 \end{bmatrix}$.

3.12

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} E(y_1) \\ E(y_2) \\ E(y_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$$

$$V(\boldsymbol{y}) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 17 \\ 6 & 6 & 12 \\ 17 & 12 & 29 \end{bmatrix}$$

(b) $E(y) = (7 + 0 + 0)/3 = 7/3$

(c) $$V(y) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 11 & 6 & 17 \\ 6 & 6 & 12 \\ 17 & 12 & 29 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = 12.8889$$

3.13

(a) H is a (nxn) symmetric matrix; $H' = H$

I - H is a (nxn) symmetric matrix; $(I - H)' = I - H$

$HH = X(X'X)^{-1}X'X(X'X)^{-1}X' = X(X'X)^{-1}X' = H$; H is idempotent

$(I - H)(I - H) = I - H - H + HH = I - H - H + H = I - H$

$HX = X(X'X)^{-1}X'X = X$

(b) $A(I - H) = (X'X)^{-1}X'(I - X(X'X)^{-1}X') = (X'X)^{-1}X' - (X'X)^{-1}X' = O$,

a (pxn) matrix of zeros

$(I - H)A' = [A(I - H)]' = O'$ a (nxp) matrix of zeros

$$H(I-H) = H - HH = H - H = O \quad \text{a (nxn) matrix of zeros}$$

$$(I-H)'H' = [H(I-H)]' = O \qquad \text{a (nxn) matrix of zeros}$$

3.14

(a) $V(A\boldsymbol{y}) = AV(\boldsymbol{y})A' = \sigma^2 AA' = \sigma^2 (X'X)^{-1}X'X(X'X)^{-1} = \sigma^2 (X'X)^{-1}$

(b) $V(H\boldsymbol{y}) = HV(\boldsymbol{y})H' = \sigma^2 HH' = \sigma^2 HH = \sigma^2 H = \sigma^2 X(X'X)^{-1}X'$

(c) $V[(I-H)\boldsymbol{y})] = (I-H)V(\boldsymbol{y})(I-H) = \sigma^2 (I-H) = \sigma^2 (I - X(X'X)^{-1}X')$

(d) $V\left(\begin{bmatrix} A \\ I-H \end{bmatrix}\boldsymbol{y}\right) = \sigma^2\left(\begin{bmatrix} A \\ I-H \end{bmatrix}\begin{bmatrix} A' & I-H \end{bmatrix}\right) = \sigma^2\begin{bmatrix} AA' & A(I-H) \\ (I-H)A' & I-H \end{bmatrix}$

$$= \sigma^2\begin{bmatrix} (X'X)^{-1} & O \\ O' & I - X(X'X)^{-1}X' \end{bmatrix}$$

3.15

(a) The eigenvalues are the solutions to the quadratic equation $|A - \lambda I| = |(1-\lambda)(1-\lambda) - \rho^2| = \lambda^2 - 2\lambda + (1-\rho^2) = 0$. They are $1+\rho$ and $1-\rho$.

(b) The eigenvector corresponding to the eigenvalue $1+\rho$ is the solution to the (vector) equation $(A - (1+\rho)I)\mathbf{p}_1 = \mathbf{0}$. The (normalized) solution is given by
$\mathbf{p}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.
Similarly, the solution to the (vector) equation $(A - (1-\rho)I)\mathbf{p}_2 = \mathbf{0}$ is given by
$\mathbf{p}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Hence $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

(c) Confirm the result by multiplication of the matrices.

(d) Experiment with several different values of ρ (-0.3, 0.3, -0.7, 0.7). Select a specific value of ρ. Use any computer software such as Minitab or SPSS to generate

20 independent random variables x_1 with variance $1+\rho$, and 20 independent random variables x_2 with variance $1-\rho$. This results in twenty independent pairs (x_1, x_2).

Apply the transformation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute the sample covariance matrix and check that it is close to the expected covariance matrix A.

3.16

(a) and (b) The steps in the derivations are spelled out in detail. Follow the algebra by substituting the relevant matrices.

(c) With correlation among the error and the regressor, the least squares estimate is no longer an unbiased estimate of β_1. This has important implications for regression modeling as standard least squares results in incorrect (biased) estimates.
Such a situation can arise if the regression model is missing an important variable, z, that is correlated with the regressor in the model, x (that is, $\rho_{\text{zx}} \neq 0$). Then the error in the incomplete original regression model can be written as $\varepsilon = \alpha \text{z} + \varepsilon^*$, where ε^* is an independent random error, and the correlation between the error and the regressor x in the model is $\rho_{\alpha \text{z} + \varepsilon^*, \text{x}} = \alpha \rho_{\text{zx}} \neq 0$.

(d) Follow the steps by using your computer software of choice for generating the random variables. For $\rho_{\varepsilon \text{x}} = 0.5$, the standard least squares estimate is estimating 2.5, and not the value $\beta_1 = 2$. For $\rho_{\varepsilon \text{x}} = -0.5$, the standard least squares estimate is estimating 1.5, and not the value $\beta_1 = 2$. The standard least squares estimate is an unbiased estimate of $\beta_1 = 2$ if $\rho_{\varepsilon \text{x}} = 0$.

3.17 The quadratic form can be written as $\boldsymbol{y}' A \boldsymbol{y}$ where the 3 x 3 symmetric matrix A is given as

$$\text{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}.$$

The determinant of this matrix is 0. The rank of the matrix A is 2, as we can find a 2x2 submatrix with a nonzero determinant. Furthermore, the matrix A is idempotent; $\text{AA} = \text{A}$. Hence the distribution of the (normalized) quadratic form

$\left(y_1^2 + 0.5y_2^2 + 0.5y_3^2 + y_2 y_3\right)/\sigma^2$ follows a chi-square distribution with 2 degrees of freedom.

3.18 The matrices in the two quadratic forms are $A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The product

$(1/\sigma^2)A_1A_2 = (1/\sigma^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence the two quadratic forms are independent.