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Multivariate contemporaneous ARMA model with hydrological applications

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Abstract: In order to allow contemporaneous autoregressive moving average (CARMA) models to be properly applied to hydrological time series, important statistical properties of the CARMA family of models are developed. For calibrating the model parameters, efficient joint estimation procedures are investigated and compared to a set of uivariate estimation procedures. It is shown that joint estimation procedures improve the efficiency of the autoregressive and moving average parameter estimates, but no improvements are expected on the estimation of the mean vector and the variance covariance matrix of the model. The effects of the different estimation procedures on the asymptotic prediction error are also considered. Finally, hydrological applications demonstrate the usefulness of the CARMA models in the field of water resources.

Key words: Contemporaneous ARMA models, maximum likelihood estimation, multivariate modelling, stochastic hydrology, time series analysis

1 Introduction

For more than two decades, hydrologists have been advocating the use of multivariate models for describing complex hydrological data. Recently, for example, the import of multivariate modeling in hydrology was reinforced by a number of manuscripts that appeared in a conference proceedings edited by Shen et al. (1986) and also a special monograph on time series analysis in water resources edited by Hipel (1985). When considering the general family of multivariate autoregressive moving average (ARMA) models, a particular subset of this family, called contemporaneous ARMA or CARMA models, is well suited for modeling hydrological time series (Salas et al. 1980; Camacho et al. 1985). The main objective of this paper is to derive useful statistical properties of CARMA models so that they can be conveniently and properly applied to hydrological, as well as other types of time series.

The contemporaneous ARMA (p,q) model, CARMA (p,q), is defined as:

$$\phi_h(B)(Z_{h,t} - \mu_h) = \theta_h(B)a_{h,t} \qquad h = 1,...,k \tag{1}$$

where $\phi_h(B) = 1 - \phi_{h1}B - \cdots - \phi_{hp_h}B^{p_h}$ is the autoregressive (AR) operator of order p_h for series h; $\theta_h(B) = 1 - \theta_{h1}B - \cdots - \theta_{hq_h}B^{q_h}$ is the moving avearge (MA) operator of order q_h for series h; $\mathbf{a}_t = (a_{1t}, ..., a_{kt})'$ is the k dimensional vector of innovations which is distributed as NID $(0, \Delta)$, where NID means normally independently distributed. Further, $\mathbf{\Delta} = (\sigma_{gh})$ is the variance-covariance matrix of $\mathbf{a}_t$, and μ_h is the mean of series $Z_{h,t}$. Also, $p = \max(p_1, \ldots, p_k)$ and $q = \max(q_1, \ldots, q_k)$.

It is assumed that the zeros of the polynomial equations $\phi_h(B) = 0$ and $\theta_h(B) = 0$, $h = 1,\ldots,k$, lie outside the unit circle so that the model is stationary and invertible, respectively. For the case where $\sigma_{gh} = 0$ for $g \neq h$ the model collapses to a set of k independent univariate ARMA (p,q) models as defined by Box and Jenkins (1976). The CARMA model describes the situation when only contemporaneous Granger causality is present among the series (see Granger 1969; Pierce and Haugh 1979 and 1977). Pierce (1977) and Hipel et al. (1985) provide empirical evidence that many economic and geophysical time series possess, in fact, only Granger instantaneous causality, so that they can be adequately fitted by CARMA models. More generally, as is pointed out by Granger and Newbold (1979), instantaneous causality may be originated when some temporal aggregation is present in the data, a situation which frequently occurs in many fields. These considerations show that the class of CARMA models is a very rich class of models and that a detailed analysis would be desirable.

Beside hydrologists, the CARMA model has been studied by workers in other fields such as statistics and economics. Nelson (1976) considers the gains in efficiency from joint estimation of CARMA model parameters. He uses bivariate AR(1) (autoregressive model of order one) and MA(1) (moving average model of order one) models in simulation experiments to illustrate such gains in efficiency and their effects on the forecasting accuracy of the model. Risager (1980, 1981), for the CAR (contemporaneous autoregressive) model, and Cipra (1984), for the bivariate CARMA model, derive the correlation structure of the model. They also provide the asymptotic distribution of the residual cross correlations. Moriarty and Salamon (1980) and Umashankar and Ledolter (1984) provide empirical evidence of the usefulness of the model to improve the forecast accuracy of the component series. For the bivariate case of the CARMA model where one series may be longer than the other, Camacho et al. (1987) develop an efficient estimation procedure which uses all of the available data.

The purpose of this paper is to give a comprehensive presentation of the statistical properties of the CARMA model. A special effort has been made to present the results as general as possible, extending in this way many of the results that have been given in the literature. For example, properties regarding the gain in efficiency in the estimation of the CARMA model have been given considering only particular models. This paper presents the general result. The effect of a joint estimation scheme on the asymptotic properties of the estimators for the mean and the variance has not been considered before. It is shown here that the asymptotic properties of the univariate estimators for the mean and the variance-covariance matrix are identical to the asymptotic properties of the corresponding joint estimators. Also, a detailed treatment of the forecast accuracy of the CARMA model is presented. Finally, some practical applications demonstrate the usefulness of the CARMA model in hydrology.

2 Estimation of parameters

The estimation of the parameters of the CARMA (p,q) model in Eq. (1) is considered in this section. To facilitate the exposition, the following notation is introduced. Let $\{\mathbf{Z}_1, \ldots, \mathbf{Z}_t\}$ where $\mathbf{Z}_t = (Z_{1t}, \ldots, Z_{kt})'$, $t = 1,\ldots,N$, be a sample of N consecutive observations from a CARMA (p,q) process. Let $\boldsymbol{\beta}_h = (\phi_{h1}, \ldots, \phi_{hp}, \theta_{h1}, \ldots, \theta_{hq})$ denote the parameters of series Z_{ht}, $h = 1,\ldots,k$, and let $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \ldots, \boldsymbol{\beta}_k')'$ denote the matrix of parameters of the CARMA model. It is assumed, without loss of generality, that the order of the univariate models are the same, i.e., $p_h = p, q_h = q, h = 1,...,k$. It is also assumed that (i) the process is stationary, (ii) invertible, (iii) $\phi_h(B)$ and $\theta_h(B)$ do not have common factors, and (iv) the innovations are Gaussian. Let $\bar{\boldsymbol{\beta}}_h$ denote the univariate maximum likelihood estimator of $\boldsymbol{\beta}_h$ obtained using the data $\{Z_{h1}, \ldots, Z_{hN}\}$. Algorithms to obtain these estimators are given elsewhere (see for example McLeod 1977; Ansley 1979; McLeod and Sales 1983). Let $\bar{\boldsymbol{\beta}} = (\bar{\boldsymbol{\beta}}_1', \ldots, \bar{\boldsymbol{\beta}}_k')'$ denote the vector of univariate estimators. The first lemma gives the asymptotic distribution of $\bar{\boldsymbol{\beta}}$.

LEMMA 1 The asymptotic distribution of $N^{1/2}(\bar{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is normal with mean vector zero and covariance matrix $\mathbf{V}_{\bar{\beta}}$

$$\mathbf{V}_{\bar{\beta}} = \begin{bmatrix} \sigma_{11}\mathbf{I}_{11}^{-1} & \cdots & \sigma_{1k}\mathbf{I}_{11}^{-1}\mathbf{I}_{1k}\mathbf{I}_{kk}^{-1} \\ \vdots & & \vdots \\ \sigma_{k1}\mathbf{I}_{kk}^{-1}\mathbf{I}_{k1}\mathbf{I}_{11}^{-1} & \cdots & \sigma_{kk}\mathbf{I}_{kk}^{-1} \end{bmatrix} \tag{2}$$

where

$$\mathbf{I}_{gh} = \begin{bmatrix} \gamma_{V_gV_h}(i-j) & \gamma_{V_gU_h}(i-j) \\ \gamma_{U_gV_h}(i-j) & \gamma_{U_gU_h}(i-j) \end{bmatrix}, \quad \gamma_{cd}(i-j) = \langle c_{t-i} \cdot d_{t-j} \rangle$$

where c, d stand for V_g, U_g, V_h, U_h, $\langle \cdot \rangle$ denotes expectation and the dimensions of $\mathbf{V}_{\bar{\beta}}$ and $\mathbf{I}_{gh}$ are $k(p+q) \times k(p+q)$ and $(p+q) \times (p+q)$, respectively. The auxiliary time series are defined by:

$$\phi_h(B)V_{ht} = -a_{ht}, \quad \text{and} \quad \theta_h(B)U_{ht} = a_{ht} \qquad h = 1,...,k \tag{3}$$

Proof: It is well known that under normality, identifiability, stationarity and invertibility conditions, the univariate ARMA model meets the usual regularity conditions for the maximum likelihood estimator (MLE) to be asymptotically normal and efficient. Therefore, the MLE $\bar{\boldsymbol{\beta}}_h$ can be expanded as;

$$\bar{\boldsymbol{\beta}}_h - \boldsymbol{\beta}_h = \sigma_{hh}\mathbf{I}_{hh}^{-1}\mathbf{S}_h + 0_p(N^{-1}) \tag{4}$$

where

$\mathbf{S}_h = (S_{h1}, \ldots, S_{h(p+q)})'$ is the score function and

$$S_{hi} = \begin{cases} -(N\sigma_{hh})^{-1} \sum_{t=1}^{N} a_{ht}V_{ht-i} & i = 1,...,p \\ -(N\sigma_{hh})^{-1} \sum_{t=1}^{N} a_{ht}U_{ht-i} & i = p+1,...,p+q \end{cases} \tag{5}$$

From Eqs. (4) and (5) it is straightforward to show that $N \langle (\bar{\boldsymbol{\beta}}_g - \boldsymbol{\beta}_g) \cdot (\bar{\boldsymbol{\beta}}_h - \boldsymbol{\beta}_h)' \rangle = \sigma_{gh}\mathbf{I}_{gg}^{-1}\mathbf{I}_{gh}\mathbf{I}_{hh}^{-1}$ which gives Eq. (2). Linear combinations of the S's are the average of Martingale differences with convergent finite variance. Therefore, normality follows from the Martingale Central limit theorem (Billingsley 1961). ■

Let $\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_1', \ldots, \hat{\boldsymbol{\beta}}_k')'$ denote the MLE of $\boldsymbol{\beta}$ using joint estimation. The following lemma gives the asymptotic distribution of $\hat{\boldsymbol{\beta}}$.

LEMMA 2 The asymptotic distribution of $N^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is normal with zero mean and variance covariance $\mathbf{V}_{\hat{\beta}}$ given by:

$$\mathbf{V}_{\hat{\beta}} = \begin{bmatrix} \sigma^{11}\mathbf{I}_{11} & \cdots & \sigma^{1k}\mathbf{I}_{1k} \\ \vdots & & \vdots \\ \sigma^{k1}\mathbf{I}_{k1} & \cdots & \sigma^{kk}\mathbf{I}_{kk} \end{bmatrix}^{-1} \tag{6}$$

where the $\mathbf{I}_{gh}$ submatrices are defined in Lemma 1 and $\mathbf{\Delta}^{-1} = (\sigma^{gh})$ is the inverse of the innovation variance covariance matrix.

Proof. It is obvious that the aforesaid assumptions (i) through (iii) of the CARMA model imply stationarity, invertibility and triangular identifiability of the model when it is considered as a multivariate ARMA model (Dunsmuir and Hannan 1976). Wilson (1973) and later Dunsmuir and Hannan, (1976) show that under such conditions $N^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is asymptotically normal with zero mean and covariance $\mathbf{I}^{-1}$ where

$$\mathbf{I} = \lim_{N\to\infty} <\partial^2\mathbf{S}/\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'> \quad \text{with} \quad \mathbf{S} = \sum_{t=1}^{N} \mathbf{a}_t'\mathbf{\Delta}^{-1}\mathbf{a}_t/2N .$$

From Eq. (1) it follows that

$$\partial\mathbf{a}_t/\partial\phi_\ell = (0,\dots,V_{ht-\ell},\dots,0)' \quad h = 1,\dots,k; \quad l = 1,\dots,p$$

$$\partial\mathbf{a}_t/\partial\theta_\ell = (0,\dots,U_{ht-\ell},\dots,0)' \quad h = 1,\dots,k; \quad l = 1,\dots,q$$

where V_{ht} and U_{ht} are the auxiliary series defined by Eq. (3). The second derivatives of $\mathbf{S}$ are given by:

$$\partial^2\mathbf{S}/\partial\beta_{gi}\partial\beta_{hj} = \frac{1}{N}\sum_{t=1}^{N}(\partial\mathbf{a}_t'/\partial\beta_{gi}\mathbf{\Delta}^{-1}\partial\mathbf{a}_t/\partial\beta_{hj} + \mathbf{a}_t'\mathbf{\Delta}^{-1}\partial^2\mathbf{a}_t/\partial\beta_{gi}\partial\beta_{hj})$$

Taking expectations, the first term of this equation becomes:

$$\sum_{t=1}^{N}\sigma^{gh}<W_{gt-i}W_{ht-j}>/N = \sigma^{gh}\gamma_{W_g W_h}(i-j)$$

where W stands for V if $\beta = \phi$ or for U if $\beta = \theta$. The second term of this equation has zero expectation. The Theorem follows by comparing this result with $\mathbf{I}_{gh}$. ■

In the following Theorem, the asymptotic distributions of $\bar{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}$ are compared. Although both estimators are asymptotically unbiased and asymptotically consistent, $\bar{\boldsymbol{\beta}}$ is not as efficient as $\hat{\boldsymbol{\beta}}$.

THEOREM 1 $\mathbf{V}_{\bar{\beta}} - \mathbf{V}_{\hat{\beta}}$ is a positive semidefinite matrix, so that $\bar{\boldsymbol{\beta}}$ is not an asymptotically efficient estimator if $\mathbf{\Delta}$ is not a diagonal matrix.

Proof. Consider the vector $\boldsymbol{\alpha} = N^{1/2}([\bar{\boldsymbol{\beta}} - \boldsymbol{\beta}]', \partial\mathbf{S}/\partial\boldsymbol{\beta}')'$, where $\mathbf{S} = \sum_{t=1}^{N}\mathbf{a}_t'\Delta^{-1}\mathbf{a}_t/2N$. Then, $\partial\mathbf{S}/\partial\beta_{hj} = \sum_{r=1}^{K}\sigma^{rh}\sum_{t=1}^{N}a_{rt}W_{ht-j}/N$, where W stands for V if $\beta = \phi$ and for U if $\beta = \theta$. Now because of the normality assumption, it follows from a well-known result of Isserlis (1918) that:

$$\begin{aligned} <a_{gt}W_{gt-i}a_{rt'}W_{ht'-j}> &= <a_{gt}W_{gt-i}><a_{rt'}W_{ht'-j}> \\ &\quad + <a_{gt}a_{rt'}><W_{gt-i}W_{ht'-j}> + <a_{gt}W_{ht'-j}><W_{gt-i}a_{rt'}> \\ &= \sigma_{gr}\gamma_{W_g W_h}(i-j)\cdot\delta(t-t') \end{aligned} \tag{7}$$

where $\delta(t) = 1$ for $t = 0$ and $\delta(t) = 0$ for $t \neq 0$.

From Eq. (5), it follows that:

$$N \langle S_{gi}\partial S/\partial\beta_{hj}\rangle = \gamma_{W_g W_h}(i-j)\cdot\sum_{r=1}^{K}\sigma^{rh}\sigma_{gr}/\sigma_{gg}$$

$$= \begin{cases} \gamma_{W_g W_h}(i-j)/\sigma_{gg} & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases}$$

This result and Eq. (4) imply that

$$N \langle (\bar{\boldsymbol\beta} - \boldsymbol\beta)\cdot\partial \mathbf{S}/\partial\boldsymbol\beta'\rangle = \mathbf{I}_{k(p+q)}$$

where $\mathbf{I}_m$ is the m-dimensional identity matrix.

Now, from Eq. (7) it is easy to show that

$$N\cdot\langle\partial\mathbf{S}/\partial\boldsymbol\beta\cdot\partial\mathbf{S}/\partial\boldsymbol\beta'\rangle = \mathbf{V}_{\hat{\boldsymbol\beta}}^{-1}\,.$$

Therefore, the variance covariance matrix of $\boldsymbol\alpha$, which is positive semidefinite, is given by:

$$\begin{bmatrix} \mathbf{V}_{\bar{\boldsymbol\beta}} & \mathbf{I} \\ \mathbf{I} & \mathbf{V}_{\hat{\boldsymbol\beta}}^{-1} \end{bmatrix}$$

It follows from a result of matrix algebra that

$$\mathbf{V}_{\bar{\boldsymbol\beta}} - \mathbf{I}\cdot(\mathbf{V}_{\hat{\boldsymbol\beta}})^{-1}\cdot\mathbf{I} = \mathbf{V}_{\bar{\boldsymbol\beta}} - \mathbf{V}_{\hat{\boldsymbol\beta}}$$

is a positive semidefinite matrix, which is the desired result. In the case that $\boldsymbol\Delta$ is a diagonal matrix, it is easy to see that $\mathbf{V}_{\bar{\boldsymbol\beta}} = \mathbf{V}_{\hat{\boldsymbol\beta}}$. ■

The next lemma provides a computationally and statistically efficient algorithm to estimate the parameters of the CARMA model.

LEMMA 3 Let $\boldsymbol\beta^* = \bar{\boldsymbol\beta} - \mathbf{V}_{\hat{\boldsymbol\beta}}(\partial\mathbf{S}/\partial\boldsymbol\beta)_{\boldsymbol\beta = \bar{\boldsymbol\beta}}$ where $\mathbf{S} = \sum_{t=1}^{N}\mathbf{a}_t'\Delta^{-1}\mathbf{a}_t/2N$. Then, $\boldsymbol\beta^*$ is an asymptotically efficient estimator.

Proof. From Lemma 1, $\bar{\boldsymbol\beta}$ is an asymptotically consistent estimator of $\boldsymbol\beta$. Therefore, $\boldsymbol\beta^*$, which corresponds to one iteration of the method of scores, has the same asymptotic properties as the MLE of $\boldsymbol\beta$ (Cox and Hinkley 1974; Harvey 1981). ■

The main idea in the above procedure is to estimate the parameters of the series, $\boldsymbol\beta_h$, $h = 1,\dots,k$, using an univariate ARMA estimation algorithm and then to calculate one iteration of the Gauss-Newton optimization scheme. Of course, iterations may be continued until convergence is obtained to give the MLE $\hat{\boldsymbol\beta}$.

The following Theorem gives the distribution of the estimators of the mean vector $\boldsymbol\mu = (\mu_1,\dots,\mu_k)$ and the variance-covariance matrix $\boldsymbol\Delta$ in the CARMA model. As before, $\bar{\boldsymbol\mu} = (\bar\mu_1, \dots, \bar\mu_k)$ denotes the vector of univariate estimators for $\boldsymbol\mu$ and $\hat{\boldsymbol\mu}$ the joint estimator. Similar notation is used for $\boldsymbol\Delta$.

THEOREM 2 The asymptotic distribution of $N^{1/2}(\bar{\boldsymbol\mu} - \boldsymbol\mu, \bar{\boldsymbol\Delta} - \boldsymbol\Delta)$ and of $N^{1/2}(\hat{\boldsymbol\mu} - \boldsymbol\mu, \hat{\boldsymbol\Delta} - \boldsymbol\Delta)$ are identical. Both are normal with zero mean and variance covariance given by:

$$\mathbf{V} = \begin{bmatrix} \mathbf{I}_\mu^{-1} & 0 \\ 0 & \mathbf{I}_\Delta^{-1} \end{bmatrix}$$

where $\mathbf{I}_\mu = [\sigma^{gh}\phi_g(1)\phi_h(1)/\theta_g(1)\theta_h(1)]$, $\mathbf{I}_\Delta = [i(\sigma_{ij},\sigma_{rs})/2]$, and $i(\sigma_{ij},\sigma_{rs}) = (\sigma^{si}\sigma^{jr} + \sigma^{sj}\sigma^{ir})/2$. Furthermore, this distribution is statistically independent of $\hat{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\beta}}$.

Proof. Consider first the distribution of $N^{1/2}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}, \hat{\boldsymbol{\Delta}} - \boldsymbol{\Delta})$. As in Lemma 2, the normality, identifiability, stationarity and invertibility conditions ensure that the regularity conditions for the asymptotic results of the MLE are satisfied. Moreover the likelihood can be approximated by (Hillmer and Tiao 1979):

$$\ell(\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{\Delta}) = C - N\log|\boldsymbol{\Delta}|/2 - \sum_{t=1}^{N} \mathbf{a}_t'\boldsymbol{\Delta}^{-1}\mathbf{a}_t/2 \tag{8}$$

It follows that the asymptotic distribution of $N^{1/2}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}, \hat{\boldsymbol{\Delta}} - \boldsymbol{\Delta})$ is normal with mean zero and variance-covariance $\mathbf{I}^{-1}$ where $\mathbf{I} = \lim_{N\to\infty} <-\partial^2\ell/\partial^2(\mu,\boldsymbol{\Delta})>/N$ is the large sample Fisher information matrix per observation. For others,

$$\partial\ell/\partial\mu_h = \sum_{t=1}^{N} a_h'\boldsymbol{\Delta}^{-1}D$$

where $D' = (0 \cdots C_h \cdots 0)$ and $C_h = -\partial a_h/\partial\mu_h = \phi_h(1)/\theta_h(1)$(see Eq. (1)). So

$$\mathbf{I}_\mu = <-\partial^2\ell/\partial^2\boldsymbol{\mu}>/N = (\sigma^{gh}C_gC_h) = \mathrm{diag}(C_1,..,C_k)\boldsymbol{\Delta}^{-1}\,\mathrm{diag}(C_1,..,C_k) \tag{9}$$

Also,

$$\partial^2\ell/\partial\sigma_{ij}\partial\mu_h = -\sum_{t=1}^{N} a_t'\Delta^{-1}\mathbf{K}_{ij}\boldsymbol{\Delta}^{-1}D = 0 + 0_p(N^{1/2}) \tag{10}$$

where $\mathbf{K}_{ij} = (K_{ij} + K_{ji})/2$ and K_{ij} is the matrix with zero entries everywhere except for a value of one in position (i,j). The last equality follows because the left side of Eq. (10) has zero expectation and variance $0(N)$. The derivatives with respect to $\boldsymbol{\Delta}$ are given by:

$$\partial\ell/\partial\sigma_{ij} = -N\sigma^{ij}/2 + tr(\boldsymbol{\Delta}^{-1}\mathbf{K}_{ij}\boldsymbol{\Delta}^{-1}\cdot\sum_{t=1}^{N}\mathbf{a}_t\mathbf{a}_t')/2 \quad \text{and}$$

$$\partial^2\ell/\partial\sigma_{ij}\partial\sigma_{rs} = N(\sigma^{ri}\sigma^{js} + \sigma^{rj}\sigma^{js})/4 - tr\{\boldsymbol{\Delta}^{-1}\mathbf{K}_{ij}\boldsymbol{\Delta}^{-1}\mathbf{K}_{rs}\boldsymbol{\Delta}^{-1} + \boldsymbol{\Delta}^{-1}\mathbf{K}_{rs}\boldsymbol{\Delta}^{-1}\mathbf{K}_{ij}\boldsymbol{\Delta}^{-1}\cdot(\sum_{t=1}^{N}\mathbf{a}_t\mathbf{a}_t')\}/2$$

Taking expectations, this becomes:

$$<\partial^2\ell/\partial\sigma_{ij}\partial\sigma_{rs}> = N(\sigma^{si}\sigma^{jr} + \sigma^{ri}\sigma^{js})/4$$

In general, $\mathbf{I}_\Delta$ can be expressed as:

$$\mathbf{I}_\Delta = <-\partial^2\ell/\partial^2\boldsymbol{\Delta}>/N = (\boldsymbol{\Delta}^{-1}\otimes\boldsymbol{\Delta}^{-1})(\mathbf{I} + P)/4$$

where P is a permutation matrix such that $P^2 = \mathbf{I}_{k^2}$ the identity matrix and $P(\boldsymbol{\Delta}^{-1}\otimes\boldsymbol{\Delta}^{-1}) = (\boldsymbol{\Delta}^{-1}\otimes\boldsymbol{\Delta}^{-1})P$. Given that $\boldsymbol{\Delta}$ is a symmetric matrix, it is only necessary to consider the $k(k+1)/2$ elements of the upper (or lower) triangular part of the matrix to obtain the Fisher information and the correlation matrices of $\boldsymbol{\Delta}$. When the k^2 elements of the matrix $\boldsymbol{\Delta}$ are considered in the calculation of the Fisher information matrix of $\boldsymbol{\Delta}$, the resulting matrix $\mathbf{I}_\Delta$ is singular because some rows of the matrix are repeated. This representation is, however, somewhat easier to work with. A generalized inverse for $\mathbf{I}_\Delta$ can be easily obtained. In fact, $\mathbf{I}_\Delta^{-1}$ can be expressed as:

$$\mathbf{I}_\Delta^{-1'} = (\mathbf{I} + P)(\boldsymbol{\Delta}\otimes\boldsymbol{\Delta}) = (\boldsymbol{\Delta}\otimes\boldsymbol{\Delta})(\mathbf{I} + P) \tag{11}$$

The result for $(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}, \hat{\boldsymbol{\Delta}} - \boldsymbol{\Delta})$ follows from Eqs. (9) to (11).

Consider now the distribution of $\bar{\boldsymbol{\mu}}$ and $\bar{\boldsymbol{\Delta}}$. As in Lemma 1, the univariate MLE $\bar{\mu}_h$ can be expanded as:

$$\bar{\mu}_h - \mu_h = I_{\mu_h}^{-1}\partial\ell_h/\partial\mu_h + 0_p(1/N) \tag{12}$$

where $\ell_h(\boldsymbol{\beta}_h,\mu_h,\sigma_{hh}) = C - N\log\sigma_{11}/2 - \sum_{t=1}^{N} a_{ht}^2/2\sigma_{hh}$ and $\mathbf{I}_\mu = \lim_{N\to\infty} <\partial^2\ell_h/\partial^2\mu_h> = NC_h^2/\sigma_{hh}$. Further,

$$N\cdot\mathrm{Cov}(\bar{\mu}_g, \bar{\mu}_h) = N<\mathbf{I}_\mu^{-1}\partial\ell_g/\partial\mu_g\cdot\mathbf{I}_\mu^{-1}\partial\ell_h/\partial\mu_h>$$
$$= (N^2C_g^2C_h^2/\sigma_{gg}\sigma_{hh})^{-1}\cdot\sum_{t=1}^{N}\sum_{t'=1}^{N}<a_{gt}a_{ht'}> = \sigma_{gh}/C_gC_h$$

The varianve-covariance matrix of $N^{1/2}(\bar{\boldsymbol{\mu}} - \boldsymbol{\mu})$ is given by

$$\mathrm{diag}(1/C_1,\ldots,1/C_k)\boldsymbol{\Delta}^{-1}\mathrm{diag}(1/C_1,\ldots,1/C_k)$$

which is the inverse of $\mathbf{I}_\mu$ given in Eq. (9). The estimators for $\bar{\boldsymbol{\Delta}}$ are given by:

$$\bar{\sigma}_{gh} = \sum_{t=1}^{N}\bar{a}_{gt}\bar{a}_{ht}/N$$

where $\bar{a}_{ht}$ are the residuals obtained from Eq. (1) using $\bar{\boldsymbol{\beta}}_h$ instead of $\boldsymbol{\beta}$, the true value. Taking a Taylor expansion around the true parameters $\boldsymbol{\beta}$, evaluating at $\bar{\boldsymbol{\beta}}$ and observing that $(\bar{\boldsymbol{\beta}} - \boldsymbol{\beta})$ and $\partial\sigma_{gh}/\partial\sigma_{hj}$ are both $0_p(N^{-1/2})$, it follows that:

$$\bar{\sigma}_{gh} = \sum_{t=1}^{N}a_{gt}a_{ht}/N + 0_p(N^{-1}) \tag{13}$$

The expectation of $\bar{\sigma}_{gh}$ is σ_{gh} and the variance-covariance of $\bar{\sigma}_{gh}$ and $\bar{\sigma}_{ij}$, neglecting terms of $0(N^{-2})$, is given by

$$<\bar{\sigma}_{ij}\cdot\bar{\sigma}_{gh}> - \sigma_{gh}\sigma_{ij} = (1/N^2)\cdot\sum_{t=1}^{N}\sum_{t'=1}^{N}<a_{it}'a_{jt}'a_{gt}a_{ht}> - \sigma_{gh}\sigma_{ij}$$
$$= (\sigma_{gi}\sigma_{jh} + \sigma_{gj}\sigma_{ih})/N$$

so that the variance covariance matrix of $N^{1/2}(\bar{\boldsymbol{\Delta}} - \boldsymbol{\Delta})$ can be written as $(\boldsymbol{\Delta}\otimes\boldsymbol{\Delta})\cdot(\mathbf{I} + P)$ which is equal to Eq. (11). Normality is obtained from the Martingale central limit theorem as in Lemma 1.

The last statement of the theorem can be proved considering the Taylor expansions of the form (see Eqs. (4), (12), (13))

$$N^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = [\mathbf{I}_{\boldsymbol{\beta}}]^{-1}\partial\ell/\partial\boldsymbol{\beta} + 0(N^{-1/2})$$

where $\ell(\cdot)$ is given by Eq. (8) and observing that

$$<\partial\ell/\partial\boldsymbol{\beta}\cdot\partial\ell/\partial(\boldsymbol{\mu}', \boldsymbol{\Delta})> = 0, \quad <\partial\ell/\partial\boldsymbol{\beta}\cdot\partial\ell/\partial\mu_h> = 0$$
$$<\partial\ell/\partial\boldsymbol{\beta}\cdot\bar{\sigma}_{gh}> = 0, \text{ and } <\partial\ell/\partial\boldsymbol{\beta}\cdot\partial\ell/\partial(\boldsymbol{\mu}', \boldsymbol{\Delta})> = 0.$$

It can also be proved that the joint distributions are normal. Because they are uncorrelated, the independence result is obtained. ■

3 Distribution of the residual autocorrelations

In this section, the large sample distribution of the residual autocorrelations for the CARMA model and an adequate Portmanteau test for the independence of the residuals is given. Li and McLeod (1981) derived the large sample distribution of the residual autocorrelations for the general multivariate ARMA model. The result for the general model is rather too complicated to be of direct applicability. For the CARMA model, a significant amount of simplification may be obtained which gives more easily applicable results.

Let $\boldsymbol{\beta}$ be a vector of parameter values satisfying conditions (i) through (iii) of Section 2. For $p + 1 < t < N$ let

$$\dot{a}_{ht} = Z_{ht} - \dot{\phi}_{h1} Z_{ht-1} - \cdots - \dot{\phi}_{hp} Z_{ht-p} + \dot{\theta}_{h1}\dot{a}_{ht-1} + \cdots + \dot{\theta}_{hq}\dot{a}_{ht-q}$$

$$\dot{a}_{ht} = 0 \quad \text{for} \quad t \leq p, \quad h = 1,\ldots,k$$

The corresponding autocorrelations are defined by:

$$\dot{r}_{gh}(\ell) = \dot{C}_{gh}(\ell)/[\dot{C}_{gg}(\ell)\dot{C}_{hh}(\ell)]^{1/2}$$

$$\dot{C}_{gh}(\ell) = \sum_{t=1}^{N-l} \dot{a}_{gt}\dot{a}_{ht+\ell}/N$$

Let also $\dot{\mathbf{r}} = (\dot{\mathbf{r}}_{11}', \dot{\mathbf{r}}_{21}', \ldots, \dot{\mathbf{r}}_{12}', \ldots, \dot{\mathbf{r}}_{kk}')'$ where $\dot{\mathbf{r}}_{ij} = [\dot{r}_{ij}(1), \ldots, \dot{r}_{ij}(M)]'$. For $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ the vector of univariate estimator (see section 2), let $\bar{a}_{ht}$ and $\bar{r}_{ij}(\ell)$ denote the corresponding residuals and residual autocorrelations. Similarly, let a_{ht}, $r_{ij}(\ell)$ and $\hat{a}_{ht}$, $\hat{r}_{ij}(\ell)$ be the residuals and the auotcorrelations corresponding to $\boldsymbol{\beta} = \boldsymbol{\beta}$, the true parameter values, and to $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$, the MLE of $\boldsymbol{\beta}$, respectively. It is also assumed through this section that $\boldsymbol{\Delta} = \langle \mathbf{a}_t \cdot \mathbf{a}_t' \rangle$ is in correlation form.

McLeod (1979) derived the distribution of the residual cross-correlation in univariate ARMA time series models. His results can be particularized to obtain the distribution of $\bar{\mathbf{r}}$. The main results for the CARMA model are summarized in the following Lemma.

LEMMA 4

(i) The asymptotic joint distribution of $N^{1/2}(\bar{\boldsymbol{\beta}} - \boldsymbol{\beta}, \mathbf{r}')$ is normal with mean zero and variance covariance

$$\begin{bmatrix} \mathbf{V}_{\bar{\beta}} & -\mathrm{diag}(\mathbf{I}_{hh}^{-1})\mathbf{A}' \\ -\mathbf{A}\mathrm{diag}(\mathbf{I}^{-1}) & \mathbf{Y} \end{bmatrix}$$

where $\mathbf{V}_{\bar{\beta}}$ and $\mathbf{I}_{hh}$ are given in Lemma 1,

$$\mathbf{Y} = \boldsymbol{\Delta} \otimes \boldsymbol{\Delta} \quad \mathbf{I}_M, \quad \mathbf{A} = (\sigma_{gh} \cdot \mathbf{X}_{hh}),$$

$$\mathbf{X}_{hh} = [\sigma_{1h} \quad \cdots \quad \sigma_{kh}]' \otimes (-\pi_{h,i-j} \mid \phi_{j,i-j})_{M_x(p+q)} = \sigma_{\cdot h} \quad \mathbf{X}_h,$$

$$\phi(B)^{-1} = \sum_{r=0}^{\infty} \pi_{hr} B^r, \quad \theta_h^{-1}(B) = \sum_{r=0}^{\infty} \psi_{hr} B^r$$

and $\mathbf{A} \otimes \mathbf{B}$ denotes the kronecker product of matrices.

(ii) The asymptotic distribution of $N^{1/2}\bar{\mathbf{r}}$ is normal with mean zero and variance-covariance

$$\mathbf{Y} + \mathbf{X}\mathbf{V}_{\bar{\beta}}\mathbf{X}' - \mathbf{X}\mathrm{diag}(\mathbf{I}_{hh}^{-1})\mathbf{A}' - \mathbf{A}\mathrm{diag}(\mathbf{I}_{hh}^{-1})\mathbf{X}'$$

where $\mathbf{X} = \mathrm{diag}(X_{II},..., X_{kk})_{k^2M\times k^2M}$.

In particular, the variance of $\bar{\mathbf{r}}_{gh} = (\bar{r}_{gh}(1), \ldots, \bar{r}_{gh}(M))$ is given by:

$$N\mathrm{Var}(\bar{r}_{gh}) = \mathbf{I}_M - \sigma_{gh}\mathbf{X}_h\mathbf{I}_{hh}^{-1}\mathbf{X}_h' \tag{14}$$

The following lemmas give the asymptotic distribution of $\hat{\mathbf{r}}$.

LEMMA 5

(i) The asymptotic joint distribution of $N^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}, r')$ is normal with mean zero and variance covariance given by

$$\begin{bmatrix} \mathbf{V}_{\hat{\beta}} & -\mathbf{V}_{\hat{\beta}}\mathbf{X}' \\ -\mathbf{X}\cdot\mathbf{V}_{\hat{\beta}} & \mathbf{Y} \end{bmatrix}$$

where $\mathbf{V}_{\hat{\beta}}$ is given by Lemma 2, $\mathbf{X}$ and $\mathbf{Y}$ by Lemma 4.

(ii) The asymptotic distribution of $N^{1/2}\hat{\mathbf{r}}$ is asymptotically normal with zero mean and covariance matrix

$$\mathbf{Y} - \mathbf{X}\cdot\mathbf{V}_{\hat{\beta}}\cdot\mathbf{X}'$$

In particular, the variance of $\hat{\mathbf{r}}_{gh} = (\hat{r}_{gh}(1),\ldots, \hat{r}_{gh}(M))$ is given by

$$N\cdot\mathrm{Var}(\hat{\mathbf{r}}_{gh}) = \mathbf{I}_M - \sigma_{gh}^2\mathbf{X}_h\cdot\mathrm{Var}(\hat{\boldsymbol{\beta}}_h)\mathbf{X}_h \tag{15}$$

A detailed proof for the lemma is given by Camacho (1984). ■

The following modified Portmanteau test statistics is useful for testing for the independence of the residuals (see Li and McLeod 1981):

$$\begin{aligned} Q_M^* &= N\hat{\mathbf{r}}'\hat{\mathbf{r}} + k^2M(M + 1)/2 \\ &= N\sum_{\ell=1}^{M}\hat{\mathbf{r}}(\ell)'(\boldsymbol{\Delta}^{-1}\otimes\hat{\boldsymbol{\Delta}}^{-1})\hat{r}(\ell) + k^2M(M + 1)/2 \end{aligned} \tag{16}$$

where $\hat{r}(\ell) = (\hat{r}_{11}(\ell), \hat{r}_{21}(\ell),\ldots, \hat{r}_{k1}(\ell), \hat{r}_{12}(\ell),\ldots, \hat{r}_{k2}(\ell),\ldots, \hat{r}_{kk}(\ell))'$ which is approximately χ^2-distributed with $k^2M - k(p+q)$ d.f. for large N and M. As shown by Li and McLeod (1981), this modified test provides a better approximation to the null distribution than $Q_M = N\cdot\hat{\mathbf{r}}'\hat{\mathbf{r}}$.

Expressions (14) and (15) also provide a method for testing the independence of the residuals by comparing the observed values of $\hat{r}_{ij}(\ell)$ or $\bar{r}_{ij}(\ell)$ with the respective asymptotic standard deviations which are easily calculated. Large values or $\hat{r}_{ij}(\ell)$ or $\bar{r}_{ij}(\ell)$, $\ell \neq 0$, should detect misspecification of the model.

4 Prediction error for the CARMA model

In this section, the effect of the two different estimation procedures, namely univariate and joint estimation, on the prediction error of the CARMA model is investigated. To begin with, it is observed that for a given set of parameter values the univariate and the joint forecasts are equal. More specifically, let $\dot{Z}_{h,t(i)}$ the ith step ahead prediction of $Z_{h,t}$ at time t using the parameter values $\dot{\boldsymbol{\beta}}$ and the univariate model be

$$\dot{\phi}_h(B)Z_{ht} = \dot{\theta}_h(B)a_{ht}$$

where it is assumed, for simplicity, that the mean of the series are equal to zero. Let $(\dot{\mathbf{Z}}_{t,i})_h$ denote the h-component of $\mathbf{Z}_{t,i}$, the ith step ahead prediction at time t

of the vector $\mathbf{Z}_t = (Z_{1t}, \ldots, Z_{kt})'$ using the model

$$\dot{\Phi}(B)\mathbf{Z}_t = \dot{\Theta}(B)\mathbf{a}_t$$

where $\dot{\Phi}(B)$ and $\dot{\Theta}(B)$ are diagonal matrices with entries $\{\dot{\phi}_1(B), \ldots, \dot{\phi}_K(B)\}$ and $\{\dot{\theta}_1(B), \ldots, \dot{\theta}_K(B)\}$, respectively. Then, it can be easily shown that $\dot{Z}_{h,t(i)} = (\dot{\mathbf{Z}}_{t,i})_h$ and $\text{Var}\{\dot{Z}_{h,t(i)}\} = \text{Var}\{(\dot{\mathbf{Z}}_{t,i})_h\}$.

The above result implies that to study the effects of different estimation procedures on the predictions of the CARMA model, it is sufficient to restrict the study to each one of the univariate models.

Bloomfield (1972) obtained the one step ahead prediction error of univariate ARMA models when the parameters of the model are estimated and showed that it depends on the estimation procedure. A more general result was given by Yamamoto (1981). He obtained formulae for the asymptotic mean square prediction error at any lag of multivariate ARMA models when the true parmaeters values are substituted for their maximum likelihood estimates. His results remain valid if a consistent estimator with asymptotic normal distribution is used instead of the maximum likelihood estimator. Yamamoto's formulae can then be exploited to obtain the prediction errors of the CARMA model under different estimation schemes. For this, let $\dot{Z}_{h,t(i)}$ denote the ith step ahead prediction of Z_{ht} using the parameter values $\dot{\boldsymbol{\beta}}$. The following lemma gives the asymptotic distribution of $\dot{Z}_{h,t(i)}$. The lemma is a straight forward modification of Theorem 2 of Yamamoto (1981, p.489). It is assumed that the observations used for forecasting are independent from those used for estimation, as is customary when dealing with asymptotic prediction errors.

LEMMA 6 Assume that $\dot{\boldsymbol{\beta}}$ is a consistent estimator for $\boldsymbol{\beta}$ with a normal asymptotic distribution with mean zero and variance-covariance $\mathbf{V}$. Then the asymptotic mean square error (AMSE) of $\dot{Z}_{h,t(i)}$ is given by

$$\text{AMSE}(\dot{Z}_{t,h(i)}) = \Omega_i + E(\mathbf{Y}_t'\mathbf{U}_i'\mathbf{V}\mathbf{U}_i\mathbf{Y}_t)$$

where $\boldsymbol{\Omega} = \mathbf{H}'\{\sum_{k=0}^{i-1}\mathbf{A}^{k-1}(\mathbf{A}-\mathbf{B})\mathbf{H}\Delta\mathbf{H}'(\mathbf{A}-\mathbf{B})'\mathbf{A}'^{k-1}\}\mathbf{H} = \sum_{k=0}^{i-1}\Psi_{hk}^2\sigma_{hh}$,

$\mathbf{Y}_t = [Z_t, Z_{t-1}, Z_{t-2}, \ldots]'$, $\mathbf{U}_i = [S_1, S_2, S_3, \ldots]$,

$$\mathbf{S} = \frac{\partial}{\partial\boldsymbol{\beta}}[\mathbf{H}'\mathbf{A}^{i-1}\mathbf{B}^r(\mathbf{A}-\mathbf{B})\mathbf{H}],$$

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & \cdots & I_{s-1} \\ \vdots & & \vdots \\ \alpha_s & \cdots & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \beta_1 & \cdots & I_{s-1} \\ \vdots & & \vdots \\ \beta_s & \cdots & 0 \end{bmatrix}, \quad \mathbf{H} = (1,0,\ldots,0)'_{s\times 1},$$

$\alpha_i = \phi_{hi}$, $i = 1,\ldots,p$, $\alpha_i = 0$, $i>p$, $\beta_j = \theta_{hj}$, $j = 1,\ldots,q$, $\beta_j = 0$, $j>q$ and $\Psi_h(B) = \theta_h(B)/\phi_h(B)$ and $s = \max(p,q)$.

The following corollary is a direct consequence of Lemma 6 and Theorem 1.

COROLLARY Let $\bar{Z}_{t,h(i)}$ and $\hat{Z}_{t,h(i)}$ denote the ith step ahead prediction error of $Z_{t,h}$ using the univariate estimated parameters $\bar{\boldsymbol{\beta}}$ and the joint estimated parameters, $\hat{\boldsymbol{\beta}}$, respectively. Then $\text{AMSE}(\bar{Z}_{h,t(i)}) \geq \text{AMSE}(\hat{Z}_{h,t(i)})$ for $i = 1,2,\ldots$.

These results can be illustrated using the bivariate CARM(1,0) process. It is well known that the asymptotic variance covariance of $\bar{\phi}_h$ is $\text{Var}(\bar{\phi}_h) = (1 - \phi_h^2)/N$ and it can be shown that

$\mathrm{Var}(\hat{\phi}_h) = \mathrm{Var}(\bar{\phi}_h)\cdot(1 - \rho^2)/(1 - a\rho^4)$ where ρ is the cross correlation at lag zero between the two processes and $a = (1 - \phi_{11}^2)(1 - \phi_{21}^2)/(1 - \phi_{11}\phi_{21})^2$. Now using Lemma 6, it follws that

$$\mathrm{AMSE}(\bar{Z}_{h,t(i)}) - \mathrm{AMSE}(\hat{Z}_{h,t(i)}) = N^{-1}\sigma_{hh}\cdot(i\phi_h^{i-1})^2\cdot\rho^2(1 - a\rho^2)/(1 - a\rho^4)$$

For the case of a bivariate CARMA(0,1), it can be shown that

$$\mathrm{AMSE}(\bar{Z}_{h,t(i)}) - \mathrm{AMSE}(\hat{Z}_{h,t(i)}) = \begin{cases} N^{-1}\sigma_{hh}\rho^2(1 - a\rho^2)(1 - a\rho^4) & i = 1 \\ 0 & i > 1 \end{cases}$$

where now $a = (1 - \theta_{11}^2)(1 - \theta_{21}^2)/(1 - \theta_{11}\theta_{21})^2$.

These examples illustrate the fact that the reduction on the forecast error obtained by using joint estimators, depends on the parameter values of the model and on the lag of the forecast.

5 Hydrological applications

As an example to show the advantages of fitting CARMA models to water resources time series, consider the first of three applications given by Camacho et al. (1985). Average annual riverflows in m^3/s for the Fox River near Berlin, Wisconsin, and the Wolf River near London, Winconsin, are available from Yevjevich (1963) and also the hydrological data tapes of Colorado State University at Fort Collins, for the years from 1899 to 1965. Because the Fox and Wolf Rivers lie within the same geographical and climatic region of North America, a priori one may expect from a physical viewpoint that a CARMA model would be more appropriate to use than separate univariate ARMA models. Subsequent to taking a natural logarithmic transformation of the observations in both time series, univariate identification results suggest that it may be adequate to fit a MA model of order one (i.e., MA(1)) to each data set. After prewhitening each series using the calibrated MA(1) model, the residual CCF (cross correlation function) for each series is calculated with the prewhitened Fox and Wolf riverflows in order to obtain the graph of the residual CCF in Fig. 1, along with the 95% confidence limits (see Haugh (1976) and Haugh and Box (1977) for a description of the residual CCF and Hipel et al. (1985) for detailed hydrological applications using the residual CCF). Because the sample residual CCF in Fig. 1 is only significantly different from zero at lag zero, this indicates that a CARMA model could be fitted to the logarithms of the bivariate series. Additionally, the fact that each series can adequately be described by a univariate MA(1) model suggests that the following CARMA(0,1) model should be used:

$$\log Z_{ht} - \mu_h = (1 - \theta_{h1}B)a_{ht} \qquad h = 1,2$$

where $h = 1$ and $h = 2$ refer to the Fox and Wolf logarithmic riverflows, respectively. Table 1 lists the parameter estimates along with their standard errors appearing in brackets, using the univariate approach (McLeod and Sales 1983) and the joint estimation algorithm developed in this paper. As can be observed in Table 1, there is a significant reduction in the variance of the parameter estimates when the joint estimation is employed. This in turn means that the relative efficiency of the univariate estimates with respect to the joint multivariate estimator is much less than unity. This relative efficiency is calculated using

$$\mathrm{eff} = \mathrm{var}(\hat{\beta}_{hi})/\mathrm{var}(\bar{\beta}_{hi})$$

where $\hat{\beta}_{hi}$ and $\bar{\beta}_{hi}$ are the joint and univariate estimates, respectively, for the parameter β_{hi}. The correlation between $\hat{a}_{1t}$ and $\hat{a}_{2t}$ is calculated to be 0.82. When the residuals of the CARMA(0,1) model are subjected to residual checking,

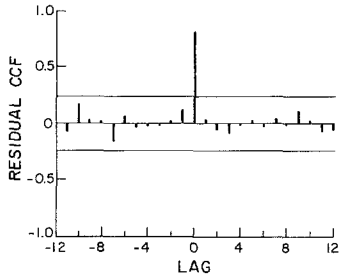

Figure 1. Residual CCF for the prewhitened series of the logarithmic Fox and Wolf riverflows

Table 1. Parameter estimates for the CARMA model and univariate models for the Fox and Wolf Rivers

	Fox River	Wolf River
Univariate Estimates of θ_{h1}	-0.483 (0.110)	-0.411 (0.111)
Joint Estimates of θ_{h1}	-0.170 (0.088)	-0.470 (0.091)
Efficiency of Univariate Estimator	0.640	0.532
Mean of Log Z_{ht}	3.39 (0.037)	3.84 (0.042)
Residual Variance	4.30×10^{-2}	7.4×10^{-2}

no misspecfications of the fitted model are detected.

In a second application, Camacho et al. (1985), fitted a CARMA model to two average monthly water quality time series. Because it is usually fairly expensive to collect water quality data, the employment of the best model at the analysis stage can be cost effective. They demonstrate that if only a univariate series were used to estimate the parameters of the model for each series, it would be necessary to increase the sample size of each series by a factor of four in order to achieve the same reduction in the variance of the parameter estimates obtained using a CARMA model.

In their final application, Camacho et al. (1985) fitted a CARMA model to two average annual riverflow time series for which one series has 70 observations while the other has 45 values. Because the estimation procedure of Camacho et al. (1987) can deal with data having unequal numbers of observations, and consequently, all of the available information can be used, their joint estimation procedure is utilized to efficiently estimate the model parameters.

Besides practical applications, simulation can also be used to demonstrate that the efficiency of the joint estimation procedure is better than the univariate estimation approach. Assuming a CARMA(1,0) model, Camacho (1984) uses simulation studies to demonstrate that for small samples the joint estimation approach is more efficient.

6 Conclusion

By employing the joint estimation procedure developed theoretically in this paper, practitioners can actually calibrate CARMA models when they fit them to hydrological and other kinds of time series. Besides theoretical results, practical applications demonstarte the usefulness of CARMA modelling in hydrology. After estimating the parameters of a CARMA model, diagnostic checking can be carried out to ensure that the CARMA model provides an adequate fit to the data set. Upon satisfying diagnostic tests, a fitted CARMA model can be used for purposes such as forecasting and simulation.

7 References

Ansley, C.F. 1979: An algorithm for the exact likelihood of a mixed autoregressive moving average process. Biometrika 66, 59-65

Bloomfield, P. 1972: On the error prediction of a time series. Biometrika 59, 501-507

Box, G.E.P.; Jenkins, G.M. 1976: Time series analysis, forecasting and control. San Francisco: Holden Day

Camacho, F. 1984: Contemporaneous CARMA modelling with application. Ph.D. thesis, Dept. of Statistical and Actuarial Sciences, The University of Western Ontario, London, Canada

Camacho, F.; McLeod, A.I.; Hipel, K.W. 1985: Contemporaneous autoregressive -- moving average (CARMA) modeling in hydrology. Water Res. Bulletin 21, 709-720

Camacho, F.; McLeod, A.I.; Hipel, K.W. 1987: Contemporaneous bivariate time series models. Biometrika 74, 103-113

Cipra, T. 1984: Simple correlated ARMA processes. Mathematische Operationsforschung und Statistik, Series Statistics 15, 513-525

Cox, D.R.; Hinkley, D.V. 1974: Theoretical statistics. London: Chapman and Hall

Dunsmuir, W.; Hannan, E.J. 1976: Vector linear time series models. Advances in App. Prob. 8, 449-464

Granger, C.W.J. 1969: Investigating casual relations by econometric models and cross spectral methods. Econometrica 37, 424-438

Granger, C.W.J.; Newbold, P. 1979: Forecasting economic time series. New York: Academic Press

Harvey, A.C. 1981: The econometric analysis of time series. Oxford: Philip Allan

Haugh, L.D. 1976: Checking the independence of two covariate stationary time series: A univariate residual cross-correlation approach. JASA 71, 378-385

Haugh, L.D.; Box, G.E.P. 1977: Identification of dynamic regression (distributed lag) models connecting two time series. JASA 72, 121-130

Hillmer, S.C.; Tiao, G.C. 1979: Likelihood function of stationary multiple autoregressive moving average models. JASA 74, 602-607

Hipel, K.W. (ed.) 1985: Time series analysis in water resources. American Water Res. Association, Bathesda, Maryland

Hipel, W.K.; McLeod, A.I.; Li, W.K. 1985: Casual and dynamic relationships between natural phenomenon. In: Anderson, O.D.; Ord, J.K.; Robinson, E.A. (eds.) Time series analysis: theory and practice 6, pp. 13-34. Amsterdam: North-Holland

Isserlis, L. 1918: On a formula for the product moment coefficient of any order of a normal frequency distribution in any number of variables. Biometrika 12, 134-139

Li, W.K.; McLeod, A.I. 1981: Distribution of the residual autocorrelations in multivariate ARMA models. J. Royal Stat. Soc. 43, 231-239

McLeod, A.I. 1977: Improved Box-Jenkins estimators. Biometrika 64, 531-534

McLeod, A.L. 1978: On the distribution of residual autocorrelations in Box-Jenkins models. J. Royal Stat. Soc. 40, 296-302

McLeod, A.I. 1979: Distribution of the residual cross correlations in univariate ARMA time series models. JASA 74, 849-855

McLeod, A.I.; Holanda Sales, P.R. 1983: Algorithm AS191: An algorithm for approximate likelihood calculation of ARMA and seasonal ARMA models. Applied Statistics 32, 211-223

Moriarty, M.; Salamon, G. 1980: Estimation and forecast performance of a multivariate time series model of sales. J. Market Research 17, 558-564

Nelson, C.R. 1976: Gains in efficiency from joint estimation of systems of autoregressive – moving average processes. J. Econometrica 4, 331-348

Pierce, D.A 1977: Relationships – and the lack thereof – between economic time series, with special reference to money and interest rates. JASA 72, 11-26

Pierce, D.A.; Haugh, L.D. 1977: Causality in temporal systems: Characterizations and survey. J. Econometrica 5, 265-293

Pierce, D.A.; Haugh, L.D. 1979: The characterization of instantaneous causality: A comment. J. Econometrica 10, 257-259

Risager, F. 1980: Simple correlated autoregressive process. Scandinarian J. Statistics 7, 49-60

Risager, F. 1981: Model checking of simple correlated autoregressive processes. Scandinarian J. Statistics 8, 137-153

Salas, J.D.; Delleur, J.W.; Yevjevich, V.; Lane, W.L. 1980: Applied modeling of hydrologic time series. Littleton, Colorado: Water Res. Publications

Shen, H.W.; Obeysekera, J.T.B.; Yevjevich, V.; Decoursey, D.G. (eds.) 1986: Multivariate analysis of hydrologic processes. Engineering Research Center, Colorado State University, Fort Collins, Colorado

Umashankar, S.; Ledolter, J. 1983: Forecasting with diagonal multiple time series models: An extension of univariate models. J. Market Research 20, 58-63

Wilson, G.T. 1973: The estimation of parameters in multivariate time series models. J. Royal Stat. Soc. 35, 76-85

Yamamoto, T. 1981: Predictions of multivariate autoregressive – moving average models. Biometrika 68, 485-492

Yevjevich, V 1963: Fluctuation of wet and dry years. I. Paper 1, Colorado State University, Fort Collins, Colorado

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