PORTMANTEAU TESTS FOR ARMA MODELS WITH INFINITE VARIANCE

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Abstract.

Autoregressive and moving-average (ARMA) models with stable Paretian errors is one of the most studied models for time series with infinite variance. Estimation methods for these models have been studied by many researchers but the problem of diagnostic checking fitted models has not been addressed. In this paper, we develop portmanteau tests for checking randomness of a time series with infinite variance and for ARMA diagnostic checking when the innovations have infinite variance. It is assumed that least-squares or an asymptotically equivalent estimation method, such as Gaussian maximum likelihood, is used. And it is assumed that the distribution of the innovations is IID stable Paretian. It is seen via simulation that the proposed portmanteau tests do not converge well to the corresponding limiting distributions for practical series length so a Monte-Carlo test is suggested. Simulation experiments show that the proposed Monte-Carlo test procedure works effectively. Two illustrative applications to actual data are provided to demonstrate that an incorrect conclusion may result if the usual portmanteau test based on the finite variance assumption is used.

Keywords. ARMA model diagnostic check, Portmanteau test, Residual autocorrelation function, Stable Paretian distribution, Testing for randomness

1. INTRODUCTION

Time series models with stable Paretian errors have been studied by many researchers. Adler et al. (1998) discussed many aspects of how to apply standard Box-Jenkins techniques to stable ARMA processes. Adler et al. (1998) concluded that, in principle, the standard Box-Jenkins techniques do carry over to the stable setting but a great deal of care needs to be exercised. In §2 we briefly review the stable Paretian distribution and in §3 we develop portmanteau tests for whiteness or randomness for an IID series. The whiteness test is illustrated with a brief application to daily returns on the S&P 500 stock index. In §4 we develop portmanteau diagnostic checks for residuals of an AR model fitted by least-squares assuming the true innovations are IID stable Paretian distributed. This is extended to the ARMA model using the equality of residuals in AR and ARMA models. An illustrative example shows the differences in inferences that may result between the finite variance and infinite variance portmanteau tests.

2. THE STABLE PARETIAN DISTRIBUTION

A stable distribution is usually defined through its characteristic function. A random variable Z, or $Z_\alpha(\sigma, \beta, \mu)$, is said to have a stable distribution if its characteristic function has the following form:

$$\mathrm{E}\left(e^{itZ}\right) = \begin{cases} \exp\left\{-\sigma|t|^\alpha \left(1 - i\beta \ \ \mathrm{sgn}(t) \tan\frac{\pi\alpha}{2}\right) + i\mu t\right\} & \text{if } \alpha \neq 1 \\ \exp\left\{-\sigma|t| \left(1 + i\beta\frac{2}{\pi} \ \ \mathrm{sgn}(t) \log|t|\right) + i\mu t\right\} & \text{if } \alpha = 1, \end{cases} \tag{1}$$

where $i^2 = -1$, t is the parameter of the characteristic function, α is the index of stability, or the characteristic exponent, satisfying $0 < \alpha \leq 2$, $\sigma > 0$ is the scale parameter, β is the skewness satisfying $-1 \leq \beta \leq 1$, $\mu \in R^1$ is the location parameter, and

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t = 0 \\ -1 & \text{if} \quad t < 0. \end{cases}$$

3. PORTMANTEAU TESTS FOR RANDOMNESS OF STABLE PARETIAN TIME SERIES

In this section, we study portmanteau tests for checking randomness of a sequence of stable Paretian random variables. We consider the stable analogues of portmanteau tests of Box and Pierce (1970) as well as Peña and Rodriguez (2002), denoted by Q_{BP} and $\hat{D}$, respectively. To do so, we require some important properties of sample autocorrelation functions (ACF) and sample partial autocorrelation functions (PACF) of stable Paretian ARMA processes (Brockwell and Davis, 1991, Ch. 13; Samorodnitsky and Taqqu, 1994; Adler et al., 1998).

3.1 Asymptotic Distribution of Sample Autocorrelation Function

Let $\{Z_t : t = 0, \pm 1, \pm 2, \ldots\}$ be an IID sequence of stable Paretian random variables and X_t be the strictly stationary process defined by

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad t = 1, \ldots, n, \tag{2}$$

where

$$\sum_{j=-\infty}^{\infty} |j| \left|\psi_j\right|^{\delta} < \infty, \quad \text{for some} \quad \delta \in (0, \alpha) \cap [0, 1]. \tag{3}$$

The stable analogue of the autocorrelation function at lag k is defined as

$$\rho_k = \sum_j \psi_j \psi_{j+k} / \sum_j \psi_j^2, \quad k = 1, 2, \ldots. \tag{4}$$

Eqn (4) can be estimated by the sample autocorrelation function as follows:

$$r_k = \left\{ \sum_{t=1}^{n-k} X_t X_{t+k} \right\} / \sum_{t=1}^{n} X_t^2, \quad k = 1, 2, \ldots, \tag{5}$$

for $\alpha > 0$. According to Davis and Resnick (1986), for any positive integer k, the limiting distribution of sample autocorrelation functions is given by

$$\left[\frac{n}{\log(n)} \right]^{\frac{1}{\alpha}} (r_1 - \rho_1, \ldots, r_k - \rho_k)^T \to (Y_1, \ldots, Y_k)^T, \tag{6}$$

where $\to$ denotes convergence in distribution and

$$Y_h = \sum_{j=1}^{\infty} (\rho_{k+j} + \rho_{k-j} - 2\rho_j \, \rho_k) \, \frac{S_j}{S_0}, \; h = 1, \ldots, k, \tag{7}$$

where $S_0, S_1, \ldots$ are independent stable variables; S_0 is positive with $S_0 \sim Z_{\alpha/2}(C_{\alpha/2}^{-2/\alpha}, 1, 0)$, and the S_j are $Z_\alpha(C_\alpha^{-1/\alpha}, 0, 0)$, where

$$C_\alpha = \begin{cases} (1-\alpha)/(\Gamma(2-\alpha)\cos(\pi\alpha/2)) & \text{if} \quad \alpha \neq 1 \\ 2/\pi & \text{if} \quad \alpha = 1. \end{cases}$$

Under the null hypothesis that X_t are a sequence of IID stable Paretian random variables, we have $\rho_0 = 1$ and $\rho_k = 0$ for $k \geq 1$ so the limiting distribution of sample ACFs can be further simplified as follows:

$$\left[\frac{n}{\log(n)} \right]^{\frac{1}{\alpha}} (r_1, \ldots, r_k)^T \to (W_1, \ldots, W_k)^T, \tag{8}$$

where W_h are given by

$$W_h = \frac{S_h}{S_0}, \; h = 1, \ldots, k. \tag{9}$$

Note that, for $\alpha > 1$, we may also use the mean-corrected sample autocorrelation function at lag k, denoted as $\tilde{r}_k$, which is given by

$$\tilde{r}_k = \sum_{t=1}^{n-k}(X_t - \bar{X})(X_{t+k} - \bar{X}) / \sum_{t=1}^{n}(X_t - \bar{X})^2, \tag{10}$$

$k = 1, 2, \ldots$. Davis and Resnick (1986) indicated that the limiting distribution of $\tilde{r}_k$ is the same as that of r_k.

3.2 Asymptotic Distribution of Sample Partial Autocorrelation Function

The sample partial autocorrelation function at lag k is defined as the sample estimate of the k-th element of the Yule-walker solution to an AR process. The sample PACF may also be calculated using the Durbin-Levison algorithm.

Let π_k be the sample PACF at lag k, and $\pi_{(m)} = (\pi_1, \ldots, \pi_m)^T$. By the Durbin-Levison algorithm, the vector $\pi_{(m)}$ can be expressed as a function of $\mathbf{r}_{(m)}$, $\pi_{(m)} = \psi(\mathbf{r}_{(m)})$, with the k-th element given by

$$\pi_k = \psi(\mathbf{r}_{(k)}) = \frac{r_k - \mathbf{r}_{(k-1)}^T \mathbf{R}_{(k-1)}^{-1} \mathbf{r}_{(k-1)}^*}{1 - \mathbf{r}_{(k-1)}^T \mathbf{R}_{(k-1)}^{-1} \mathbf{r}_{(k-1)}}, \tag{11}$$

where $\mathbf{r}_{(p)} = (r_1, \ldots, r_p)^T$ is the $p \times 1$ vector of sample autocorrelation functions, $\mathbf{R}_{(p)} = (r_{|i-j|})_{p \times p}$ is the $p \times p$ sample autocorrelation matrix and $\mathbf{r}_{(k)}^* = (r_k, \ldots, r_1)^T$.

Following the proof in Monti (1994), we can derive the asymptotic distribution of sample partial autocorrelation functions. Under the null hypothesis that X_t are independent, the autocorrelation functions are all zero, and according to Brockwell and Davis (1991, ch. 13),

$$r_h = O_p\left(\left[\frac{n}{\log(n)}\right]^{-1/\alpha}\right), \quad h = 1, 2, \ldots.$$

Therefore,

$$\mathbf{R}_{(k)} = \mathbf{1}_k + O_p\left(\left[\frac{n}{\log(n)}\right]^{-1/\alpha}\right),$$

where $\mathbf{1}_k$ is a $k \times k$ identity matrix. By eqn. (11),

$$\pi_{(m)} = \mathbf{r}_{(m)} + O_p\left(\left[\frac{n}{\log(n)}\right]^{-2/\alpha}\right). \tag{12}$$

Using eqn. (8), we have

$$\left[\frac{n}{\log(n)}\right]^{\frac{1}{\alpha}} (\pi_1, \ldots, \pi_m)^T \to (W_1, \ldots, W_m)^T. \tag{13}$$

3.3 Asymptotic Distributions of Q_{BP} and $\hat{D}$ Tests

Under the assumption that $1 < \alpha < 2$, Runde (1997) derived the limiting distribution of Q_{BP}, based on the mean corrected sample autocorrelation functions. His result is given by

$$Q_{\text{BP}}(m) = \left(\frac{n}{\log(n)}\right)^{2/\alpha} \sum_{j=1}^{m} \tilde{r}_j^2 \to W_1^2 + \cdots + W_m^2, \tag{14}$$

where $\{W_k : k = 1, \ldots, m\}$ are defined in eqn. (9). Note that if $0 < \alpha \leq 1$, the limiting distribution of eqn. (14) remains the same if $\tilde{r}_k$ are replaced by r_k.

Consider next the $\hat{D}$ test of Peña and Rodriguez (2002). In the stable case we may define the test statistic,

$$\hat{D}(m) = \left(\frac{n}{\log(n)}\right)^{2/\alpha} \left(1 - |\mathbf{R}_{(m)}|^{1/m}\right). \tag{15}$$

Using the results in §3.1 and §3.2, and following the arguments of Peña and Rodriguez (2002), we may have the asymptotic distribution of eqn. (15) in the following Theorem.

THEOREM 1 *$\hat{D}(m)$ in eqn. (15) is asymptotically distributed as*

$$\sum_{i=1}^{m} \frac{m+1-i}{m} W_i^2,$$

where $\{W_i : i = 1, \ldots, m\}$ are as defined in eqn. (9).

The proof of this theorem is given in Appendix A.

Remark 1: The limiting distributions of the Q_{BP} and $\hat{D}$ tests can be computed by making use of the change variable technique and some numerical algorithms of calculating the probability density function of stable random variables, such as Mittnik et al. (1999). This approach requires, however, intensive numerical computations.

Remark 2: Another approach to obtaining the asymptotic distributions of the Q_{BP} and $\hat{D}$ tests is to simulate the aforementioned tests based on their asymptotic distributions. For example, $\hat{D}$ is simulated as defined in Theorem 1. This approach also requires lengthy computations but it is much less intensive computationally than the approach mentioned in Remark 1. This approach will be adopted in the subsequent analysis based on 10^4 simulations.

Remark 3: Peňa and Rodriguez (2006) consider a slightly different normalization for the $\hat{D}$ statistic and an improved finite-sample approximation to its limiting distribution is provided. In the stable case, the new test statistic may be written,

$$\hat{D}^*(m) = -\left(\frac{n}{\log(n)}\right)^{2/\alpha} |\mathbf{R}_{(m)}|^{1/m}. \tag{16}$$

It may be shown that the limiting distribution of $\hat{D}^*(m)$ is as stated in Theorem 1. Also identical results are produced if $\hat{D}^*(m)$ is used instead of $\hat{D}(m)$ in the Monte-Carlo test given in Appendix B.

3.4 Simulation Experiments

Based on 250 simulations, the 5, 10, 30, 50, 70, 90, 95, 97.5, 99 (%) empirical quantiles of both $Q_{\mathrm{BP}}(m)$ and $\hat{D}(m)$ tests with lag $m = 5$ were calculated and plotted against the corresponding asymptotic distributions as stated in §3.3. It is seen in Figure 1 and Figure 2 that the empirical and asymptotic quantiles do not agree very well unless n is very large. The practical applications of both tests would be severely impeded by the slow convergence. A solution to this problem is to use the Monte-Carlo test or parametric bootstrap.

[Figures 1 and 2 about here]

Consider the simulation experiments. IID random sequence of $Z_\alpha(1, 0, 0)$ with series length $n = 250$ and $\alpha = 1.9, 1.7, 1.5, 1.3, 1.1$ were simulated. The empirical sizes of both tests were calculated based on $N = 10^4$ simulations and each Monte-Carlo test was simulated based on $B = 10^3$ simulations – see Appendix B for algorithm details. The results are tabulated in Table 1. It is seen that the empirical sizes of both tests are very close to the 5% nominal level even with $n = 250$.

[Table 1 about here]

3.5 Illustrative Example

The daily S & P 500 stock index from January 2, 1999 to December 29, 2006 was obtained from Wharton Data Research Services. This results in a series with length $n = 2011$. The returns, $\log(z_{t+1}/z_t)$, were computed and tested for randomness. The consistent estimators of McCulloch (1986) were used to estimate α and β for the returns. We obtained $\hat{\alpha}_M = 1.587$ and $\hat{\beta}_M = -0.081$. It is seen that $\hat{\beta}_M$ is close to zero so the series is not highly skewed. Since $\hat{\alpha}_M$ is much less than 2, the Monte-Carlo test using the stable distribution should be used (Appendix B). For comparison, the Monte-Carlo tests were also done using normal random variables as well. The Ljung-Box tests based on the χ^2 approximation are shown in Table 2. The Ljung-Box test (Ljung and Box, 1976) rejects at level α when $Q_{\text{LB}} > \chi^2_m(1-\alpha)$, where

$$Q_{\text{LB}}(m) = n(n+2)\sum_{k=1}^{m} r_k^2/(n-k), \tag{17}$$

and $\chi^2_m(1-\alpha)$ denotes the $1-\alpha$ quantile of a χ^2 distribution with m degrees-of-freedom. As expected the Ljung-Box test agrees very well the Monte-Carlo test using the Box-Pierce statistic Q_{BP} for the Monte-Carlo test using normal random variables. There is a striking difference in the stable/normal Monte-Carlo tests when $m = 50$. Under the normality assumption both $\hat{D}$ and Q_{BP} indicate that the randomness assumption is strongly rejected whereas the P-value is only about 5% when the stable assumption is made.

[Table 2 about here]

Remark 4: Portmanteau tests based on the nonparametric bootstrap procedure could also be used but it would be expected that they would be less powerful since less information is used. In addition, the proposed parametric bootstrap test procedure can be used to test nonlinear time series driven by stable Paretian innovations.

4. DIAGNOSTIC CHECK FOR MODEL ADEQUACY OF AR(p) MODELS WITH STABLE PARETIAN ERRORS

4.1 Some Asymptotic Results

In this section, we consider Q_{BP} and $\hat{D}$ tests for diagnostic checks in model adequacy of AR(p) models with stable Paretian errors.

Define the general AR(p) process as follows:

$$\phi(B)X_t = Z_t, \tag{18}$$

where $\{Z_t : t = 0, \pm 1, \pm 2, \ldots\}$ is an IID sequence of stable Paretian random variables, B denotes the backward operator, and $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$. Let $\hat{\phi}_{(p)} = (\hat{\phi}_1, \ldots, \hat{\phi}_p)^T$ denote the estimates of autoregressive coefficients. The residuals of the fitted model are given as follows:

$$\hat{Z}_t = Z_t(\hat{\phi}_{(p)}) = X_t - \hat{\phi}_1 X_{t-1} - \ldots - \hat{\phi}_p X_{t-p} = \hat{\phi}(B)X_t, \tag{19}$$

and the corresponding residual autocorrelation at lag k is given by

$$\hat{r}_k = \frac{\sum \hat{Z}_t \hat{Z}_{t-k}}{\sum \hat{Z}_t^2}.$$

Consider the estimators of $\hat{\phi}_{(p)}$ satisfying

$$\hat{\phi}_{(p)} = \phi_{(p)} + O_p\left([n/\log(n)]^{-1/\alpha}\right).$$

From Lin and McLeod (2007, Appendix B) the residual autocorrelation at lag k, $\hat{r}_k$, can be approximated by the first order Taylor expansion about error autocorrelation functions, r_k. Specifically, the approximation is

$$\hat{r}_k = r_k + \sum_{j=1}^{p} (\phi_j - \hat{\phi}_j)\, \psi_{k-j} + O_p\left([n/\log(n)]^{-2/\alpha} \right), \tag{20}$$

where ψ_j is the impulse response coefficient at lag j and $r_k = \sum Z_t Z_{t-k} / \sum Z_t^2$ is the error autocorrelation at lag k. Eqn. (20) can also be written in matrix form, to order $O_p\left([n/\log(n)]^{-2/\alpha} \right)$,

$$\hat{\mathbf{r}}_{(p)} = \mathbf{r}_{(p)} + \mathbf{X}\left(\phi_{(p)} - \hat{\phi}_{(p)} \right), \tag{21}$$

where

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \psi_1 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \psi_{m-1} & \psi_{m-2} & \cdots & \psi_{m-p} \end{bmatrix}. \tag{22}$$

By making use of eqn. (20) or eqn. (21) as well as following the proof in Peňa and Rodriguez (2002), we may derive the asymptotic distributions of the aforementioned portmanteau tests for diagnostic checks in AR(p) models. This distribution, however, is usually very complicated and may not be traceable unless the AR(p) models of interest are fitted by least squares (LS). For simplicity, therefore, we only consider the case that eqn. (18) is estimated using least squares in the subsequent analysis.

According to §4 in Davis (1996), if the ARMA parameters, β, are estimated using least squares , we have $[n/\log(n)]^{1/\alpha}\left(\hat{\beta}_{LS} - \beta \right)$ converges in distribution, where $\hat{\beta}_{LS}$ denotes the LS estimates of β. Hence, in terms of our notation, we have $\hat{\phi}_{(p)} - \phi_{(p)} = O_p\left([n/\log(n)]^{-1/\alpha}\right)$. Then, by Box and Pierce (1970), $\{\hat{Z}_t\}$ in eqn. (19) satisfy the orthogonality conditions and, to order $O_p\left(1/\sqrt{n}\ [n/\log(n)]^{-1/\alpha}\right)$,

$$\hat{\mathbf{r}}_{(p)}^T \mathbf{X} = 0. \tag{23}$$

If we now multiply eqn. (21) on both sizes by

$$\mathbf{Q} = \mathbf{X}(\mathbf{X}^T\ \mathbf{X})^{-1}\mathbf{X}^T,$$

then using eqn. (23) we have

$$\hat{\mathbf{r}}_{(p)} = (\mathbf{1}_m - \mathbf{Q})\,\mathbf{r}_{(p)} \tag{24}$$

approximately, where $\mathbf{1}_m$ is an $m \times m$ identity matrix and $\mathbf{Q} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$. It was shown by Box and Pierce (1970) that $\mathbf{1}_m - \mathbf{Q}$ is idempotent of rank $m - p$. Hence, the asymptotic distribution of the Q_{BP} test is given by

$$(\frac{n}{\log n})^{2/\alpha} \sum_1^m \hat{r}_k^2 \to \mathbf{W}_m^T(\mathbf{1}_m - \mathbf{Q})\mathbf{W}_m, \tag{25}$$

where $\mathbf{W}_m = (W_1, \ldots, W_m)^T$ and $\{W_i : i = 1, \ldots, m\}$ are defined in eqn. (9).

Consider next the asymptotic distributions of residual partial autocorrelations. Let $\hat{\pi}_{(m)}$ be the vector of the first m residual partial autocorrelations and $\pi_{(m)}$ is the vector of error partial autocorrelations. The Taylor expansion of $\psi(\hat{\mathbf{r}}_{(m)})$ around $\mathbf{r}_{(m)}$ yields

$$\hat{\pi}_{(m)} = \pi_{(m)} + \frac{\partial \pi_{(m)}}{\partial \mathbf{r}_{(m)}}\left(\hat{\mathbf{r}}_{(m)} - \mathbf{r}_{(m)}\right) + O_p\left(\left[\frac{n}{\log n}\right]^{-2/\alpha}\right). \tag{26}$$

By eqn. (11) and (12), eqn. (26) becomes

$$\hat{\pi}_{(m)} = \hat{\mathbf{r}}_{(m)} + O_p\left(\left[\frac{n}{\log n}\right]^{-2/\alpha}\right). \tag{27}$$

Consider the Peña-Rodriguez test as the form of

$$\hat{D} = (\frac{n}{\log n})^{2/\alpha}\left(1 - |\hat{\mathbf{R}}_{(m)}|^{1/m}\right), \tag{28}$$

where $\hat{\mathbf{R}}_{(m)} = (\hat{r}_{|i-j|})_{m,m}$ is the $m \times m$ residual autocorrelation matrix. By eqn. (27) and following the proof in Peña and Rodriguez (2002), the limiting distribution of eqn. (28) is $\mathbf{W}_m^T\,\mathbf{A}_m\,\mathbf{W}_m$, where $\mathbf{A}_m = (\mathbf{1}_m - \mathbf{Q})^T\,\mathcal{W}_{m,m}\,(\mathbf{1}_m - \mathbf{Q})$ and $\mathcal{W}_{m,m}$ is a $m \times m$ diagonal matrix with (i,i)-th element equal to $(m-i+1)/m$ for $i = 1, \cdots, m$.

Remark 5: It is shown in Lin and McLeod (2007, Appendix B.4) that the residuals in a fitted ARMA model are asymptotically equivalent to those in a particular AR model. Hence the asymptotic results for the AR may be extended to the ARMA case.

4.2 Some Size and Power Calculations

As in §3.4, the slow convergence of Q_{BP} and $\hat{D}$ tests to their asymptotic distributions is also present at the residual autocorrelations. The first order autoregressive process $X_t = 0.5X_{t-1} + Z_t$ with $Z_t \sim Z_{1.2}(1,0,0)$ was simulated and AR (1) models were fitted to the data. Then the 5, 10, 30, 50, 70, 90, 95, 97.5, 99 (%) empirical quantiles of $\hat{r}_1$ were compared with the corresponding asymptotic quantiles. It is seen from Figure 3 that the empirical quantiles of $\hat{r}_1$ get closer to the asymptotic ones as n increases. The slow convergence of residual autocorrelations to its asymptotic distribution may cause difficulties in using portmanteau tests in practice. Therefore, as in §3.4, we suggested using the Monte-Carlo test to improve the effectiveness of portmanteau tests.

[Figure 3]

We now investigate the effectiveness of Q_{BP} and $\hat{D}$ tests for diagnostic check in fitted AR models with stable Paretian errors. The empirical sizes of $\hat{D}$ and Q_{BP} tests for a 5% significance test were first calculated via simulation. In this experiment, AR(1) models, $X_t = \phi_1 X_{t-1} + Z_t$, were simulated, where $Z_t \sim Z_{1.5}(1, 0, 0)$ and $\phi_1 = 0, \pm 0.1, \pm 0.3, \pm 0.5, \pm 0.7, \pm 0.9$ and AR(1) models were fitted to the simulated data by the Burg algorithm. The empirical size for each test was calculated based on $N = 10^4$ simulations and each Monte Carlo test used 10^3 simulations. Series length $n = 100$ and lags $m = 5, 10, 20$ were investigated. It is seen in Table 3 that the empirical sizes of both tests are very close to their nominal level.

[Table 3]

The empirical powers of $\hat{D}$ and Q_{BP} tests as diagnostic tools were also investigated via simulation. Twelve ARMA(2, 2) models of series length $n = 100$ in Table 4 of Peňa and Rodriguez (2002) were simulated and AR(1) models were fitted to the simulated data using the Burg algorithm. Both tests with lags $m = 5, 10, 20$ were calculated using the parametric bootstrap procedure. The empirical powers were calculated based on $N = 10^3$ simulations and each Monte Carlo test used 10^3 simulations. It is seen in Table 4 that the empirical powers of both tests are reasonably good for most models. Some of them are even better than the powers listed in Peňa and Rodriguez (2002). In addition, increasing the series length can also improve the effectiveness of the proposed test procedure. For example, with model 3 in Table 2, if the series length was increased to $n = 250$, the empirical powers of the $\hat{D}$ test at lags $m = 5, 10, 20$ were increased significantly from 23.37%, 20.10% and 17.61% to 58.27%, 43.71% and 35.52%, respectively. Similar improvement was also found in the Q_{BP} test. Finally, as in Peňa and Rodriguez (2002), our simulation experiments show that $\hat{D}$ is more powerful than Q_{BP} as a diagnostic tool.

[Table 4]

Remark 6: It is well known that the Burg estimate of ϕ_1 is close to the LS estimate. The advantage of using Burg estimate is that it is always in the stationary region and this is needed for the Monte-Carlo test.

4.3 Illustrative Application

Consider the the monthly simple returns of CRSP value-weighted index from January 1926 to December 1997, $n = 864$ (Tsay, 2002, Ch. 2). After fitting an AR(5), we obtained $\hat{\alpha} = 1.635$ using the method of McCulloch (1986). So the infinite variance hypothesis is plausible for this data. In Table 5 we compare the P-values obtained for the Monte-Carlo tests with $\hat{D}$ and Q_{BP} for both the stable and normal cases with $m = 10, 20, 30$. The P-value for the Q_{LB} test using the χ^2 approximation is also shown. As expected the Monte-Carlo test using Q_{BP} agrees well with the Q_{LB} test using the χ^2 approximation since n is quite large. It is interesting that when $m = 10$ all tests have similar P-values but when $m = 20, 30$ the P-values of the normal based tests are much smaller, by a factor of about 10, than the more correct test based on the stable distribution. In general it appears that using tests based on the assumption of normally distributed innovations may produce P-values which are too small when the innovations are generated by a stable distribution. In other words, the test based on the normal distribution would error in the opposite direction as a conservative test and lead to an inflated Type I error rate. We may conclude that in fitting ARMA time series with stable innovations, it is important to use the Monte-Carlo diagnostic test presented in this paper.

[Table 5]

5. CONCLUDING REMARKS

McLeod and Li (1984) considered using a portmanteau diagnostic test based on the squared residuals. Li (2004, Ch.5) shows that this test is also useful for testing for the presence of conditional heteroscedasticity in the residuals of fitted ARMA models. The Monte-Carlo test presented in this article is generalized to the squared residuals case in Appendix B and implemented in our R package (Lin and McLeod, 2007).

APPENDIX

A. Proof of Theorem 1

First, by decomposing the determinant of the sample autocorrelation matrix $\mathbf{R}_{(m)}$, Pena and Rodriguez (2002) showed that $|\mathbf{R}_{(m)}|^{1/m}$ is a weighted function of the first m partial autocorrelations. Specifically,

$$|\mathbf{R}_{(m)}|^{1/m} = \prod_{i=1}^{m}(1-\pi_i^2)^{(m+1-i)/m}. \tag{29}$$

Suppose that under the null hypothesis, $\hat{D}$ is asymptotic distributed as $\mathcal{X}$. By applying the δ-method to $g(x) = \log(1-x)$, it follows that $-\left(n/\log(n)\right)^{2/\alpha} \log\left(|\mathbf{R}_{(m)}|^{1/m}\right)$ is asymptotically distributed as $\mathcal{X}$. From eqn. (29), we can have

$$\begin{aligned} &- \left(\frac{n}{\log(n)}\right)^{2/\alpha} \log\left(|\mathbf{R}_m|^{1/m}\right) = \\ &- \left(\frac{n}{\log(n)}\right)^{2/\alpha} \sum_{i=1}^{m} \frac{m-i+1}{m} \log(1-\pi_i^2). \end{aligned} \tag{30}$$

Next suppose that

$$\left(\frac{n}{\log(n)}\right)^{2/\alpha} \left(\pi_1^2, \pi_2^2, \ldots, \pi_m^2\right)^T \longrightarrow Y, \tag{31}$$

and apply the multivariate δ-method to

$$g(\pi_1^2, \pi_2^2, \ldots, \pi_m^2) = -\sum_{i=1}^{m} \frac{m-i+1}{m} \log(1-\pi_i^2),$$

it follows that

$$-\sum_{i=1}^{m} \frac{m-i+1}{m} \log(1-\pi_i^2) \rightarrow \left(1, \frac{m-1}{m}, \ldots, \frac{1}{m}\right) Y. \tag{32}$$

From the Cramer-Wold theorem, it follows that

$$\begin{aligned} \left(1, \frac{m-1}{m}, \cdots, \frac{1}{m}\right) \left(\left(\frac{n}{\log(n)}\right)^{2/\alpha} \pi_1^2, \ldots, \left(\frac{n}{\log(n)}\right)^{2/\alpha} \pi_m^2 \right)^T \\ \longrightarrow \left(1, \frac{m-1}{m}, \ldots, \frac{1}{m}\right) Y \end{aligned} \tag{33}$$

Under the null hypothesis that X_t are a sequence of IID stable Paretian random variables, we have $\rho_0 = 1$ and $\rho_k = 0$ for $k \geq 1$ so the limiting distribution of sample ACFs can be further simplified as follows:

$$\left[\frac{n}{\log(n)}\right]^{\frac{1}{\alpha}} (r_1, \ldots, r_k)^T \rightarrow (W_1, \ldots, W_k)^T, \tag{34}$$

where W_h are given by

$$W_h = \frac{S_h}{S_0},\ h = 1, \ldots, k. \tag{35}$$

Note that, for $\alpha > 1$, we may also use the mean-corrected sample autocorrelation function at lag k, denoted as $\tilde{r}_k$, which is given by

$$\tilde{r}_k = \sum_{t=1}^{n-k}(X_t - \bar{X})(X_{t+k} - \bar{X}) / \sum_{t=1}^{n}(X_t - \bar{X})^2, \tag{36}$$

$k = 1, 2, \ldots$.

Using eqn. (8), we have

$$\left[\frac{n}{\log(n)}\right]^{\frac{1}{\alpha}} (\pi_1, \ldots, \pi_m)^T \rightarrow (W_1, \ldots, W_m)^T. \tag{37}$$

it follows that

$$\left(1, \frac{m-1}{m}, \ldots, \frac{1}{m}\right)\left(\left(\frac{n}{\log(n)}\right)^{2/\alpha}\pi_1^2, \ldots, \left(\frac{n}{\log(n)}\right)^{2/\alpha}\pi_m^2\right)^T \longrightarrow W_1^2 + \frac{m-1}{m}W_2^2 + \ldots + \frac{1}{m}W_m^2, \tag{38}$$

Finally, from eqn. (33) and eqn. (38),

$$\left(1, \frac{m-1}{m}, \ldots, \frac{1}{m}\right) Y \rightarrow \sum_{i=1}^{m} \frac{m+1-i}{m} W_i^2,$$

and from (31), we have the

$$\hat{D} \rightarrow \sum_{i=1}^{m} \frac{m+1-i}{m} W_i^2. \qquad \square$$

B. Monte-Carlo Test for Randomness and Goodness-of-Fit of ARMA Models

The Monte-Carlo test procedure for diagnostic checking of AR and ARMA models with stable Paretian errors can be summarized below.

Step 1 Fit an AR model to data using least-squares or the Burg algorithm or for ARMA, an approximate Gaussian maximum likelihood algorithm is used. Calculate residuals $\{\hat{Z}_t\}$ and the portmanteau test of interest , say $\hat{D}_m$.

Step 2 Estimate the parameters of the stable distribution from residuals $\{\hat{Z}_t\}$ in Step 1. The quantile estimator given by McCulloch (1986) may be used or the MLE implemented in the R package `fBasics`.

Step 3 Select the number of Monte-Carlo simulations, B. Typically $100 \leq B \leq 1000$.

Step 4 Simulate the fitted model using the estimated AR or ARMA parameters in Step 1 and $\hat{\alpha}$ in Step 2. Obtain $\hat{D}_m$ after estimating the parameters in the simulated series.

Step 5 Repeat Step 4 B times counting the number of times k that a value of $\hat{D}_m$ greater than or equal to that in Step 1 has been obtained.

Step 6 The P-value for the test is $(k+1)/(B+1)$.

Step 7 Reject the null hypothesis if the P-value is smaller than a predetermined significance level.

This algorithm is easily modified to handle the problem of testing a time series for randomness as well as for Monte-Carlo testing when the normality assumption is made.

Simulation of ARMA Models with Stable Innovations

The stable parameters may be estimated using the method of McCulloch (1986). IID stable random variables may be simulated using the `fBasics` package. The ARMA (p, q) model may be approximated as a high-order moving-average model,

$$Y_t \doteq Z_t + \psi_1 Z_{t-1} + \ldots + \psi_Q Z_{t-Q}. \tag{39}$$

Initial values $Y_1, \ldots, Y_r, r \geq \max(p, q)$ are obtained from eq. (39) using the Fast Fourier transform to compute the convolution. The remaining portion of the time series may be computed recursively. For speed, these recursions are implemented in C and interfaced to our R functions. The function `SimulateARMA` implements these ideas in our R package.

Nonlinear Test

In Steps 1 and 4 of the algorithm in Appendix A, square the residuals, compute the mean-corrected sample autocorrelations and use these autocorrelations to compute the test statistic $\hat{D}_m$.

R Package

The R package `PRTest` that implements all methods described above is available from our online supplement (Lin and McLeod, 2007). Scripts are also given (Lin and McLeod, 2007) for generating all figures and tables which were given in our paper.

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TABLE I. EMPIRICAL SIZES (%) OF MONTE-CARLO TESTS FOR A NOMINAL 5% TEST FOR RANDOMNESS OF STABLE WHITE NOISE INNOVATIONS WITH INDEX α. THE EMPIRICAL SIZE FOR EACH TEST WAS CALCULATED BASED ON $N = 10^4$ SIMULATIONS. EACH MONTE CARLO TEST USED $B = 10^3$ SIMULATIONS. SERIES LENGTH $n = 250$ AND LAGS $m = 5, 10, 15$ WERE INVESTIGATED.

α	$\hat{D}$			Q_{BP}		
	$m = 5$	$m = 10$	$m = 15$	$m = 5$	$m = 10$	$m = 15$
1.9	5.30	4.66	4.78	4.96	4.71	4.87
1.7	5.18	4.44	4.44	4.82	4.43	4.41
1.5	4.82	4.99	5.13	5.07	5.27	5.30
1.3	4.80	5.03	5.18	5.04	5.00	5.27
1.1	5.26	5.33	5.12	5.33	5.25	5.15

Table II. P-values using Monte-Carlo portmanteau tests and Ljung-Box χ^2 test, Q_{LB}, for testing randomness of returns of S&P 500 stock index.

Test Statistic	Method	$m = 5$	$m = 10$	$m = 20$	$m = 50$
$\hat{D}$	Monte-Carlo/Stable	0.104	0.087	0.068	0.049
$\hat{D}$	Monte-Carlo/Normal	0.151	0.117	0.050	0.006
Q_{BP}	Monte-Carlo/Stable	0.078	0.136	0.053	0.048
Q_{BP}	Monte-Carlo/Normal	0.107	0.237	0.019	0.005
Q_{LB}	χ^2	0.112	0.251	0.018	0.003

TABLE III. EMPIRICAL SIZES (%) OF $\hat{D}$ AND Q_{BP} FOR A 5% SIGNIFICANCE TEST. $\hat{D}$ AND Q_{BP} TESTS FOR CHECKING MODEL ADEQUACY OF AR(1) MODELS FITTED BY THE BURG ALGORITHM. BOTH TESTS WERE IMPLEMENTED BY THE PARAMETRIC BOOTSTRAP PROCEDURE. THE EMPIRICAL SIZE FOR EACH TEST WAS CALCULATED BASED ON $N = 10^4$ SIMULATIONS. EACH MONTE CARLO TEST ALSO USED $B = 10^3$ SIMULATIONS. SERIES LENGTH $n = 100$ AND LAGS $m = 5, 10, 20$ WERE INVESTIGATED.

ϕ_1	$m = 5$		$m = 10$		$m = 20$	
	$\hat{D}$	Q_{BP}	$\hat{D}$	Q_{BP}	$\hat{D}$	Q_{BP}
0.9	4.90	4.60	4.75	4.71	4.88	4.96
0.7	4.97	4.95	5.20	4.94	5.16	5.42
0.5	5.37	5.55	5.32	5.12	5.14	5.16
0.3	5.11	5.13	4.90	4.80	4.82	5.26
0.1	4.92	5.14	5.01	4.75	5.20	4.86
−0.1	5.30	5.25	5.45	5.08	5.29	4.90
−0.3	5.00	4.79	5.20	5.30	5.33	5.45
−0.5	5.00	5.00	4.93	4.93	5.10	5.26
−0.7	5.62	5.20	5.73	5.45	5.65	5.41
−0.9	5.21	5.01	5.02	5.00	5.07	5.30

TABLE IV. EMPIRICAL POWERS (%) OF $\hat{D}$ AND Q_{BP} FOR A 5% SIGNIFICANCE TEST. $\hat{D}$ AND Q_{BP} TESTS FOR CHECKING MODEL ADEQUACY OF TWELVE ARMA$(2,2)$ MODELS IN TABLE 3 OF PEÑA AND RODRIGUEZ (2002) FITTED BY AR(1) USING THE BURG ALGORITHM. BOTH TESTS WERE IMPLEMENTED BASED ON THE PARAMETRIC BOOTSTRAP PROCEDURE. THE EMPIRICAL POWER FOR EACH TEST WAS CALCULATED BASED ON $N = 10^4$ SIMULATIONS. EACH MONTE CARLO TEST ALSO USED $B = 10^3$ SIMULATIONS. SERIES LENGTH $n = 100$ AND LAGS $m = 5, 10, 20$ WERE INVESTIGATED.

Model	$m = 5$		$m = 10$		$m = 20$	
	$\hat{D}$	Q_{BP}	$\hat{D}$	Q_{BP}	$\hat{D}$	Q_{BP}
1	53.32	29.59	38.31	21.76	32.77	19.25
2	99.01	94.53	98.56	70.46	98.01	59.61
3	23.37	21.62	20.10	16.71	17.61	15.17
4	77.13	60.82	59.38	40.29	48.12	35.15
5	93.22	84.66	87.62	66.68	79.84	58.46
6	13.74	10.68	11.17	9.13	10.05	8.61
7	26.51	17.56	26.25	13.80	24.92	13.05
8	33.92	27.36	26.68	20.60	23.57	19.25
9	99.44	98.71	99.27	93.17	99.16	78.88
10	76.71	40.62	58.06	28.39	48.50	25.94
11	99.01	94.02	98.46	67.04	97.87	57.11
12	99.89	99.86	99.87	99.63	99.48	99.48

Table V. An example using the monthly simple return of CRSP value-weighted index data from Tsay (2002). The data were fitted by an AR(5) model. The entries in the first two columns are the P-values, in percent, of $\hat{D}^S$ and $\hat{Q}^S$ in §4 based on the Monte-Carlo test; those in the third and fourth columns are the P-values, in percent, of the Monte-Carlo test for $\hat{D}^N$ and $\hat{Q}^N$ based on the normal distribution and in the last column the P-value for the standard asymptotic Ljung-Box test based on the normal assumption is given.

Test Statistic	Method	$m = 10$	$m = 20$	$m = 30$
$\hat{D}$	Monte-Carlo/Stable	5.1	3.1	1.7
Q_{BP}	Monte-Carlo/Stable	5.0	2.5	2.6
$\hat{D}$	Monte-Carlo/Normal	6.6	1.4	0.1
Q_{BP}	Monte-Carlo/Normal	4.8	0.2	0.2
Q_{LB}	χ^2 test	4.7	0.2	0.2

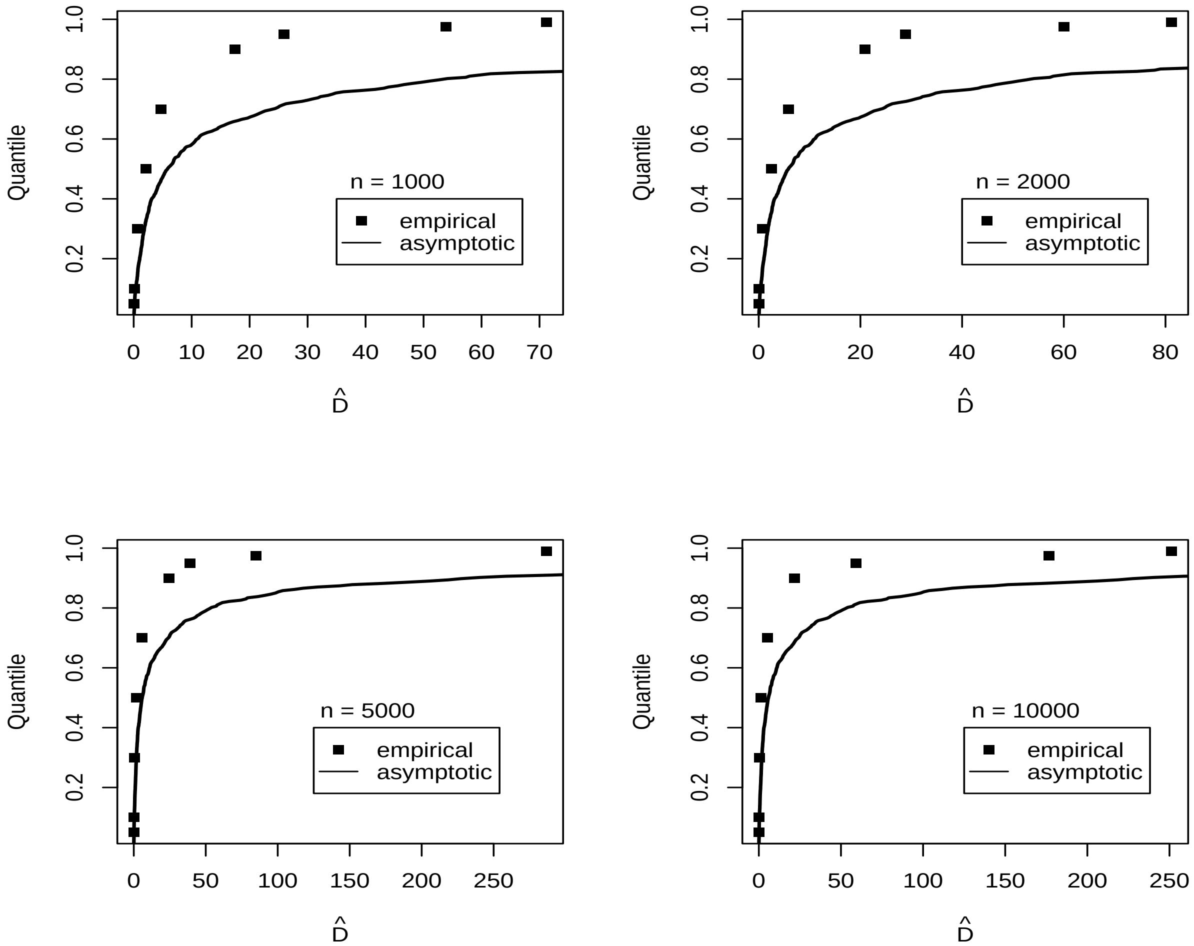

Figure 1: Quantile plot comparing the empirical and asymptotic distributions of $\hat{D}(m), m = 5$ defined in eqn. (theddhat). Random sequences of series length $n = 1000, 2000, 5000, 10000$ were simulated from $S_\alpha(1, 0, 0), \alpha = 1.2$ and empirical quantiles corresponding to 5, 10, 30, 50, 70, 90, 95, 97.5, 99 (%) are shown.

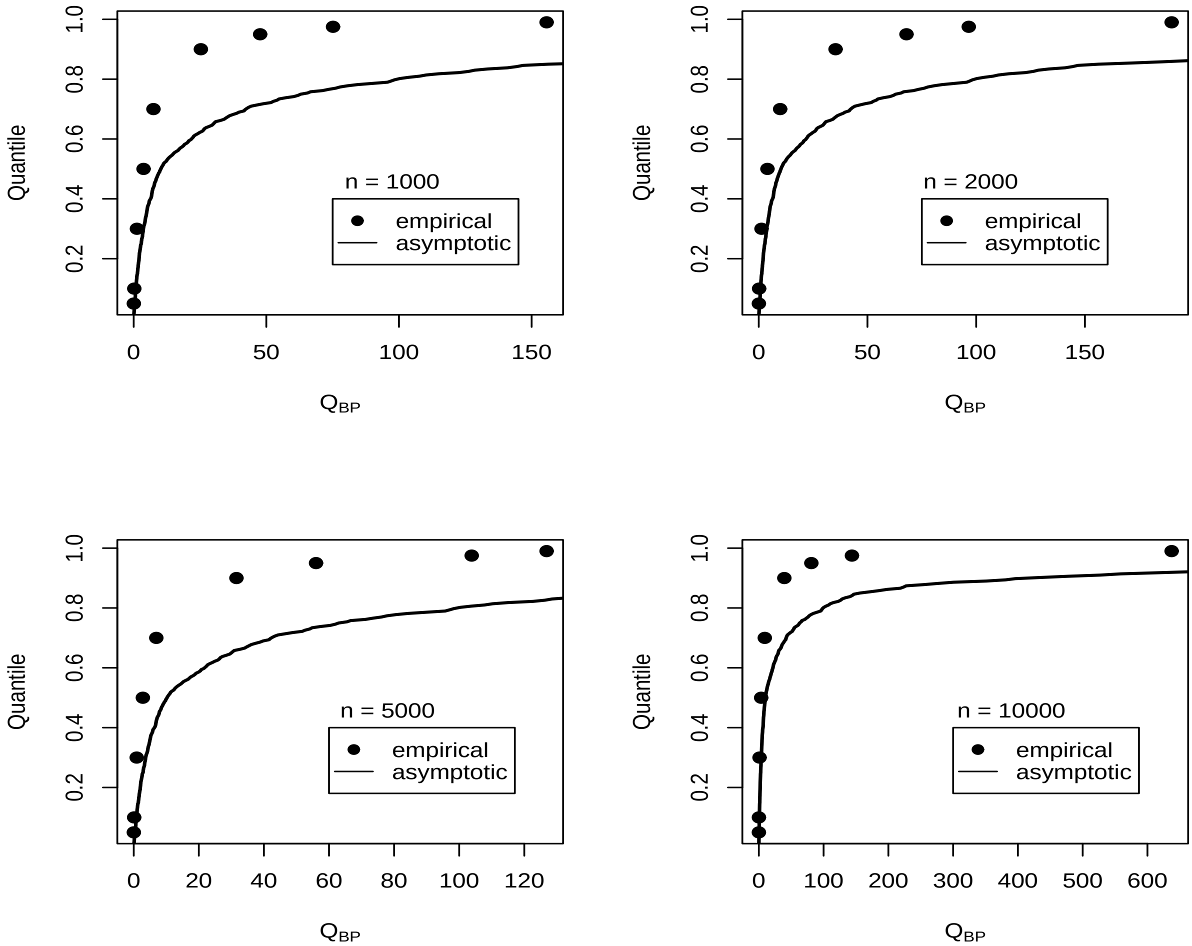

Figure 2: Quantile plot comparing the empirical and asymptotic distributions of $Q_{\mathrm{BP}}(m), m = 5$ defined in eqn. (theQBP). Random sequences of series length $n = 1000, 2000, 5000, 10000$ were simulated from $S_\alpha(1, 0, 0), \alpha = 1.2$ and empirical quantiles corresponding to 5, 10, 30, 50, 70, 90, 95, 97.5, 99 (%) are shown.

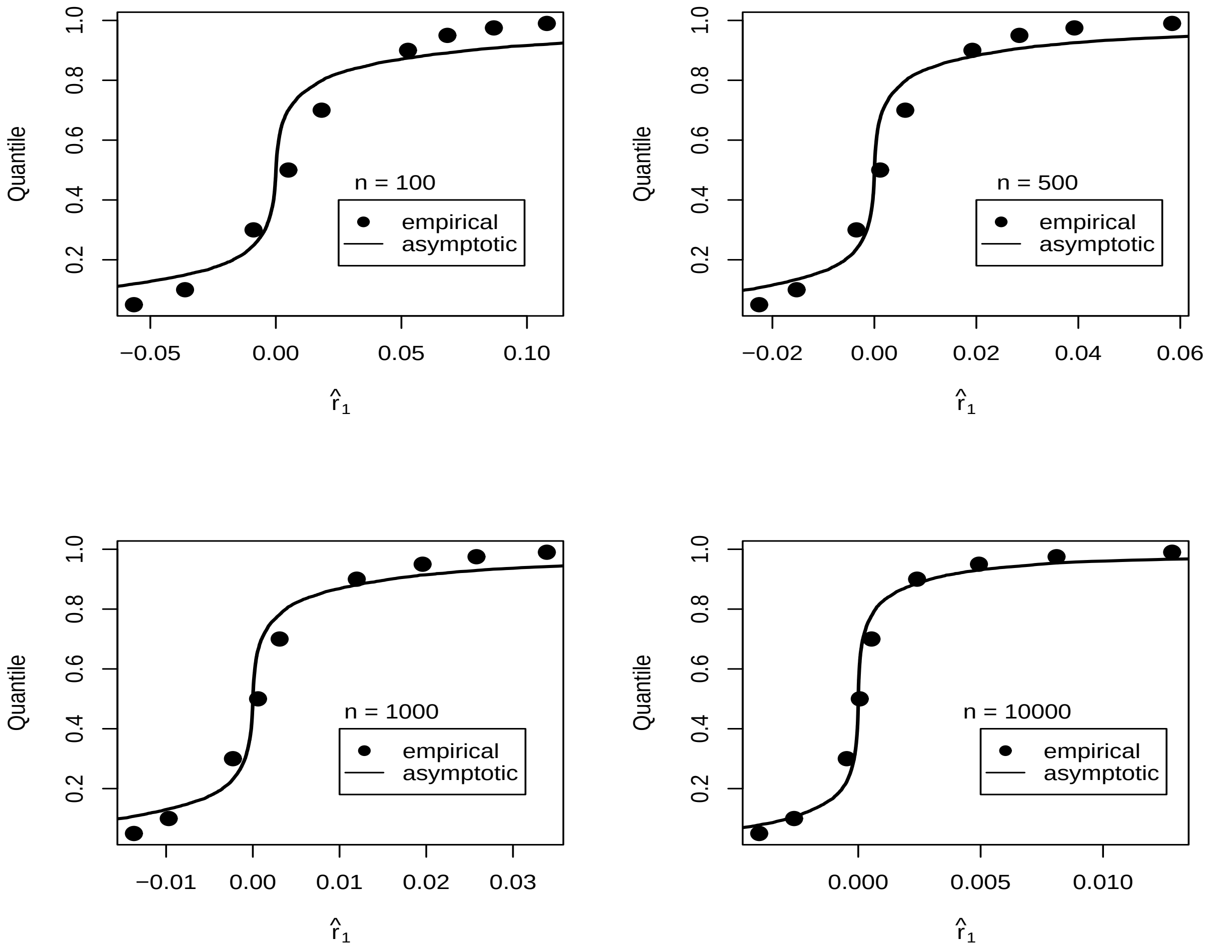

Figure 3: Quantile plot $\hat{r}_1$ and its asymptotic approximation. The AR (1) process, $X_t = 0.5X_{t-1}+Z_t$, of series length n = 100, 500, 10000 was simulated with stable innovations, $\{Z_t\} \sim S_{1.2}(1,0,0)$. For each n, 10^4 simulations were done and AR (1) models were then fitted to simulated series. The 5, 10, 30, 50, 70, 90, 95, 97.5, 99 (%) empirical quantiles of $\hat{r}_1$ are plotted and the corresponding asymptotic distribution is shown.