A Note On ARMA Model Parameter Redundancy

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Abstract

A simple condition, which is expressed directly in terms of the ARMA model parameters, is given for determining ARMA model redundancy.

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The stationary and invertible ARMA(p, q) model, which may be written in operator notation as,

$$\phi(B)z_t = \theta(B)a_t,$$

where, $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p, \theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q, a_t$ is white noise with variance σ_a^2 and B is the backshift operator on t, is said to be not redundant if and only if $\phi(B) = 0$ and $\theta(B) = 0$ have no common roots. Due to stationarity and invertibility, all roots of the equation $\phi(B)\theta(B) = 0$ are assumed to be outside the unit circle. As shown in Box and Jenkins (1975, p.240) the large sample Fisher information matrix per observation may be written,

$$I(\phi, \theta) = \frac{1}{\sigma_a^2}\mathrm{E}\{A_t A_t^T\},$$

where $A_t = (v_{t-1}, \ldots, v_{t-p}, u_{t-1}, \ldots, u_{t-q}), \quad \phi(B)v_t = -a_t, \quad \theta(B)u_t = a_t$ and the superscript T denotes the transpose. Poskitt and Tremayne (1981) showed that the information matrix $I(\phi, \theta)$ is nonsingular if and only if the model is not redundant and that the number of common roots is equal to the nullity of $I(\phi, \theta)$. In fact, this condition holds for a much simpler matrix J.

The auxiliary matrix, J, of an ARMA(p, q) model is defined by

$$J = \begin{array}{c} \\ p+q \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad p & \quad\; q \end{array} \\ \begin{pmatrix} \theta_{i-j} & -\phi_{i-j} \end{pmatrix} \end{array},$$

where, $\phi_0 = \theta_0 = -1, \quad \phi_i = 0 \quad (i < 0 \text{ or } i > p), \quad \theta_i = 0 \quad (i < 0 \text{ or } i > q)$ and the (i, j) entry in each partitioned matrix is indicated. The matrix J plays an important role in the duality theory for ARMA time series models (McLeod, 1984).

The autoregressive adjoint of the ARMA(p, q) model is defined by

$$\phi^*(B)w_t = a_t,$$

where,

$$\begin{aligned} \phi^*(B) &= \phi(B)\theta(B), \\ &= 1 - \phi_1^* B - \ldots - \phi_{p+q}^*. \end{aligned}$$

Then from McLeod (1984),

$$I(\phi, \theta) = J I(\phi^*) J^T.$$

Since $I(\phi^*)$ is positive definite, the result follows from the result of Poskitt and Tremayne (1981) and Sylvester's law of conservation.

Simple algebraic conditions which give a necessary and sufficient condition for lack of model redundancy can be derived by setting the determinant of J not equal to zero. This is easily implemented using the *Mathematica* programming language (Wolfram, 1988). Using *Mathematica*, the following condition was derived for the ARMA(2,2) model:

$${\phi_2}^2 + \phi_1\phi_2\theta_1 - \phi_2{\theta_1}^2 - {\phi_1}^2\theta_2 - 2\phi_2\theta_2 + \phi_1\theta_1\theta_2 + {\theta_2}^2 \neq 0.$$

It should be noted that the result given in this paper is mostly of theoretical interest. In practice, when fitting models to data, model redundancy is indicated when the covariance matrix of the parameters is nearly singular or the estimates of the standard deviations of the parameters are quite large. Exact model redundancy of the type discussed in this note does not arise when fitting models to data due to the uncertainty in the parameter values.

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