THE MATHEMATICS OF EXCESS OF LOSS COVERAGES AND RETROSPECTIVE RATING—A GRAPHICAL APPROACH

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Abstract

The mathematics of excess of loss coverages and retrospective rating involves heavy algebra, mainly because the indemnity payment under such contracts assumes different functional forms in different parts of the loss size distribution. This paper presents a graphical approach to the theory, in which the indemnity payment under various conditions is represented by the regions in a graph described by the cumulative distribution function of the size of loss. Many intricate formulas and relations occurring in the two subjects, some expressible algebraically only in very complicated forms, can be understood simply and clearly through pictures. Treated visually in this paper are many mathematical relations and results included in the examination syllabus.

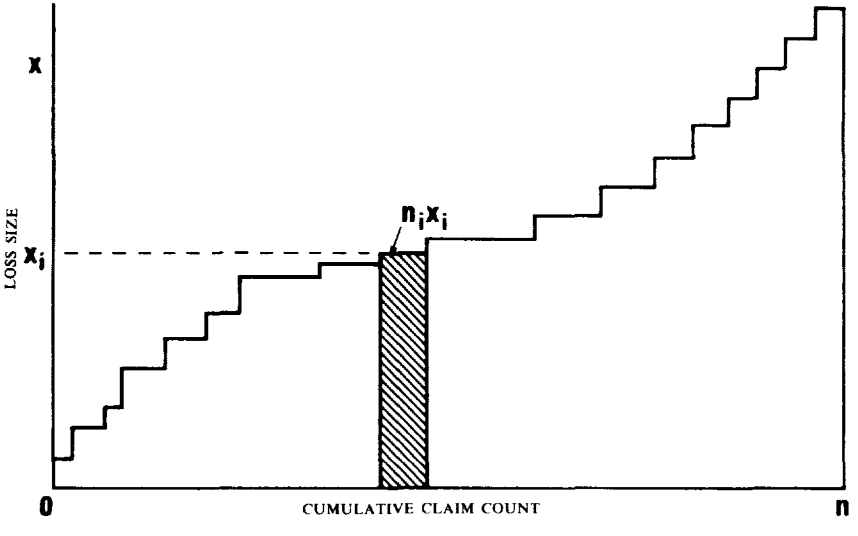
1. INTRODUCTION

The theory of excess of loss coverages and retrospective rating involves rather complicated mathematics. The underlying ideas in most cases are relatively simple, but the heavy algebra is often a great mental burden to the actuary and the student. This paper applies a graphical technique to excess of loss coverages and retrospective rating. Most of the algebraic results on these topics can be interpreted in graphic terms. The advantages of this approach are that the results so derived are easier to understand and the formulas can be easily remembered and written down.

Graphical methods are widely used in mathematics and statistics to visually present ideas which would otherwise be abstruse. Many mathematical ideas have geometric as well as symbolic interpretation. For example, the integral of a positive-valued function can be regarded as the area under the curve representing the function as well as the anti-derivative of the function. The use of diagrams and graphs to present numerical information in statistics is better known. Graphs in statistics are used to explain ideas such as density functions and cumulative distribution functions. In actuarial science, graphical methods have not been extensively utilized. A graphical device is presented herein for the explanation of the underlying mathematical ideas. It will not only provide powerful insight into the abstract relations, but also make the mathematical procedure much easier to follow compared with algebraic manipulations. For those who always prefer algebra, it will serve at least as a very useful supplement to the predominantly algebraic treatment that has been given to the subject in the literature.

To start with, consider a large number of losses, of sizes x_1, x_2, . . . , x_k, occurring n_1, n_2, . . . , n_k times, respectively, with $n = n_1 + \ldots + n_k$. In Figure 1 we represent these losses by means of a cumulative frequency curve, in which the ordinate represents the loss size, and the abscissa represents the cumulative number of losses, $c_i = n_1 + \ldots + n_i$, $i \leq k$. This representation is different from the usual form in statistical textbooks, where the abscissa and ordinate are reversed, but agrees with the representation in Snader [10]. (See also Philbrick [7].)

FIGURE 1
A CUMULATIVE FREQUENCY CURVE



The curve is a step function (with argument along the vertical axis) which has a jump of n_i at the point x_i. Consider the shaded vertical strip in the graph. It has an area equal to $n_i x_i$. Summing all such vertical strips we have

$$\text{Total amount of loss} = n_1 x_1 + \ . \ . \ . + n_k x_k. \tag{1.1}$$

We may therefore interpret the area of the vertical strip corresponding to x_i as the amount of loss of size x_i, and the total enclosed area below the cumulative frequency curve as the total amount of loss. In fact, we have a new way of viewing the cumulative frequency function curve. This curve can be constructed by arranging the losses in ascending order of magnitude, and laying them from left to right with each loss occupying a unit horizontal length.

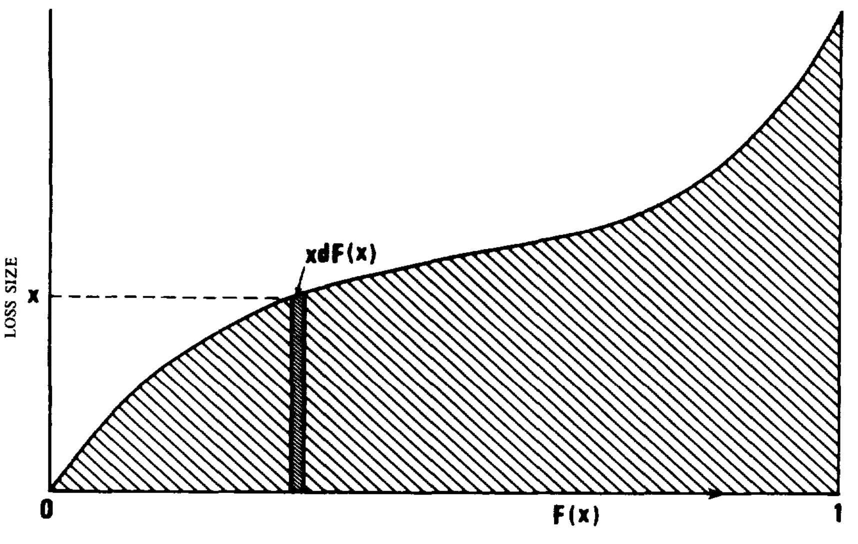
Now let X be a random variable representing the amount of loss incurred by a risk. Define the cumulative distribution function (cdf) $F(x)$ as

$$F(x) = \Pr(X \leq x). \tag{1.2}$$

Figure 2 shows the graph of a continuous cdf. Consider the vertical strip

FIGURE 2

CDF CURVE AND EXPECTATION



in the graph, with area $xdF(x)$. If we sum up all these strips, we will obtain the expected value of X,

$$E(X) = \int_0^\infty xdF(x), \tag{1.3}$$

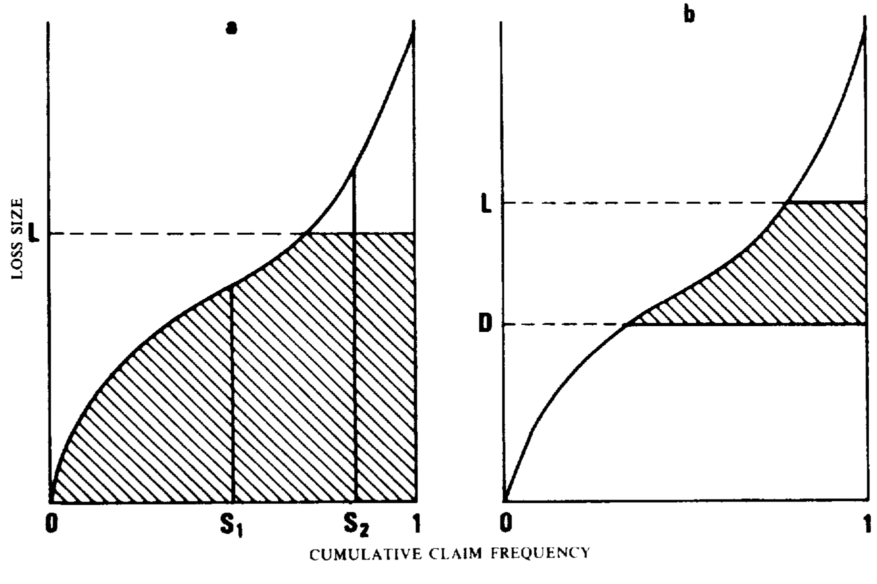
which is represented by the enclosed area below the cdf curve (the shaded area in the graph). We may interpret the expected loss as composed of losses of different sizes, and the strip $xdF(x)$ as the contribution from losses of size between x and $x+dx$. Throughout this paper, an expression such as $E\{X\}$ represents the expected value of a random variable X.

Limited Payments

As an immediate application, consider a coverage which pays for losses up to a limit L only. Figure 3(a) shows that a loss of size not more than L, such as S_1, is paid in full, while a loss of size S_2 which is greater than L, is paid only an amount L. By summing up vertical strips as before, except that strips with length greater than L are limited to length L, we obtain the expected payment per loss under such a coverage as the shaded area in Figure 3(a).

FIGURE 3

EXPECTED LOSS WITH (a) LIMIT AND (b) DEDUCTIBLE



Deductibles

Likewise, a coverage which pays for losses subject to a flat deductible D and up to limit L has expected payment per loss represented by the shaded area in Figure 3(b).

Size and Layer

As another application we first derive an integration identity. Consider Figure 4(a). The vertical strip has area $xdF\ (x)$ and the horizontal strip has area $G\ (x)\ dx$, where

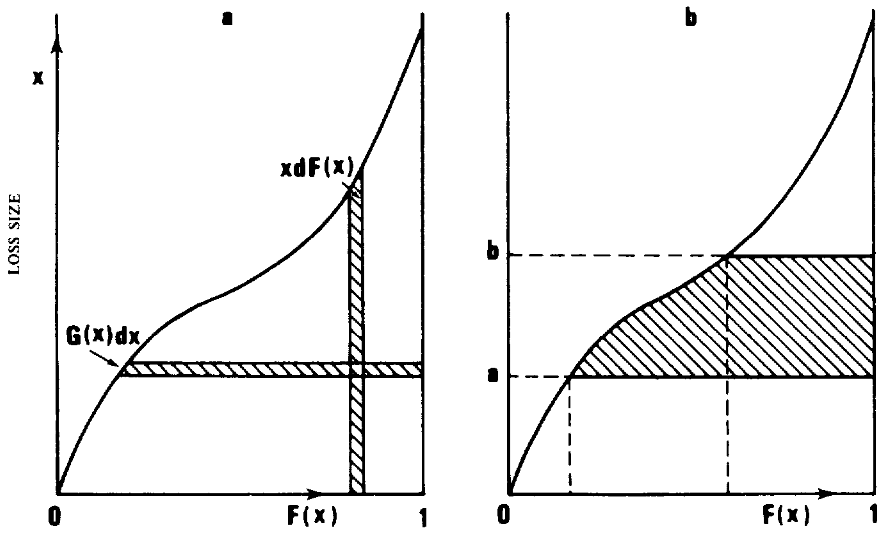
$$G\ (x) = 1 - F\ (x). \tag{1.4}$$

Summing up the vertical strips and the horizontal strips separately we have

$$\int_0^{\infty} xdF\ (x) = \int_0^{\infty} G\ (x)\ dx = E\ \{X\}, \tag{1.5}$$

because each of the integrals is equal to the enclosed area below the cdf curve, which, as we have seen, also represents the expected loss $E\ \{X\}$. The equality can also be algebraically derived via integration by parts.

FIGURE 4
SIZE AND LAYER VIEWS OF LOSSES



The two modes of summation correspond, in fact, to two views of the losses. The vertical strips group losses by size, whereas the horizontal strips group the loss amounts by layer. We may therefore call them the size method and the layer method. It is often more convenient to evaluate the expected loss in a layer by layer fashion, i.e. summing horizontal strips, than by the size method, i.e. summing vertical strips. For example, consider the layer of loss between a and b in Figure 4(b). The expected loss in this layer is represented by the shaded area. The layer method of summation gives simply

$$\int_a^b G(x)\,dx. \tag{1.6}$$

To express this integral by the size method is more difficult. A moment's reflection, with the help of Figure 4(b), yields the following expression for the integral:

$$\int_a^b x\,dF(x) + bG(b) - aG(a). \tag{1.7}$$

Again, the equality of the two expressions can be established via integration by parts.

The more complicated expression derived from the size method is the form commonly found in the literature. Although the integral associated with the layer method is simple in form, $G(x)$ is a function that is generally more difficult to integrate. This disadvantage disappears, however, when the distribution is given numerically, as, for example, when actual experience is used. The retrospective rating Table M and Table L have been constructed by the layer method; see Simon [8] and Skurnick [9]. We shall give the graphical interpretation later.

2. EXPECTED VALUE PREMIUM

Generally, given a loss X, a coverage would pay an amount depending on the value of X. We may represent this function by $g(X)$. The expected payment per loss is

$$E\{g(X)\} = \int_0^\infty g(x)\,dF(x). \tag{2.1}$$

The number of losses incurred by a risk in a policy period is a random variable, N, so that the total loss payment is

$$Y = \sum_{i=1}^{N} g\ (X_i), \tag{2.2}$$

which is the sum of a random number of random variables. It is customarily assumed that the loss severity X is distributed independently of the loss frequency N. With this assumption it can be shown that the expected payment in a policy period is

$$E\ \{Y\} = E\ \{N\} \cdot E\ \{g\ (X)\}, \tag{2.3}$$

which says that the expected value pure premium of a risk is the product of average frequency of loss and the average severity. (See Miccolis [5].)

Increased Limits Coverage

A liability insurance coverage is generally written to cover a loss in full up to a specified maximum dollar amount for any one loss. Let k be such a policy limit. We can express the payment function $g\ (X;\ k)$ of a loss X as

$$g\ (X;\ k) = \begin{cases} X, & 0 < X \le k \\ k & k < X \end{cases} \tag{2.4}$$

The expected payment per loss under this coverage can be expressed as

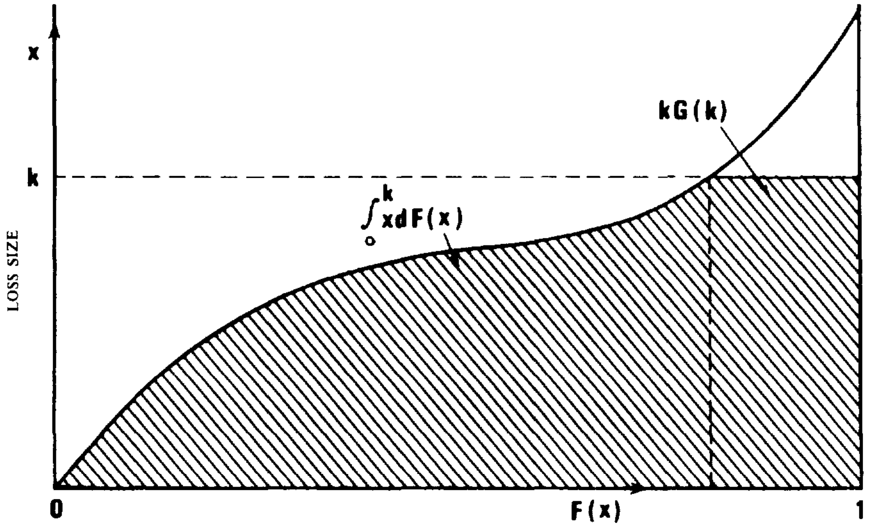
$$E\ \{g\ (X;\ k)\} = \int_0^k x dF\ (x) + kG\ (k). \tag{2.5}$$

The formula is demonstrated graphically in Figure 5, where the integral on the right is represented by the shaded area to the left of the broken vertical line, while the term $kG\ (k)$ is represented simply by the rectangle to the right of the line.

Rates are generally published for some standard limit called the basic limit; let this be b. Increased limits rates are expressed as a factor, $I\ (k)$, called the increased limits factor, to be applied to the basic limits pure premium rate. Thus

$$\begin{aligned} I\ (k) &= [E\ \{g\ (X;\ k)\} \cdot E\ \{N\}]\ /\ [E\ \{g\ (X;\ b)\} \cdot E\ \{N\}] \\ &= E\ \{g\ (X;\ k)\}\ /\ E\ \{g\ (X;\ b)\}, \end{aligned} \tag{2.6}$$

FIGURE 5
LOSSES WITH INDEMNITY LIMITED TO k



which depends on the distribution of size of loss only; see Miccolis [5]. The situation is demonstrated in Figure 6, where the increased limits factor is the ratio of the shaded area up to k versus the shaded area up to b. The picture also displays another property of the increased limits factor. Miccolis [5] shows that the derivative of $I(k)$ can be expressed as

$$I'(k) = G(k) / E\{g(X; b)\}. \quad (2.7)$$

The picture shows that when k is increased by dk, the area representing the expected payment is increased by $G(k)\,dk$. Hence the result shown in Figure 6.

Miccolis also discusses a consistency test for increased limits factors. A picture will provide much better insight into this question. In Figure 7, the enclosed region below the cdf curve is divided into horizontal panels which, for convenience of exposition, have equal width. The horizontal lines serve to subdivide a loss, such as L, into layers. With layers of equal width, the picture makes it quite plain that the expected payment in any layer is less than that in a preceding layer. If the layers

FIGURE 6
INCREASED LIMITS FACTOR

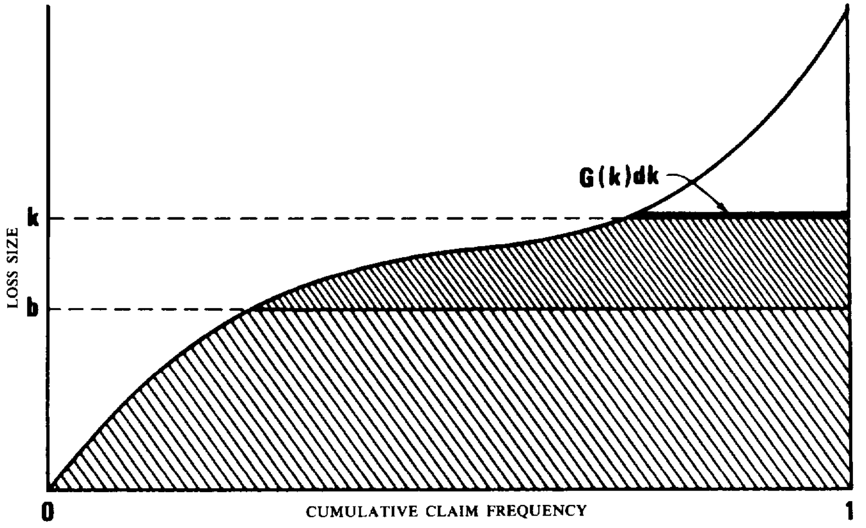
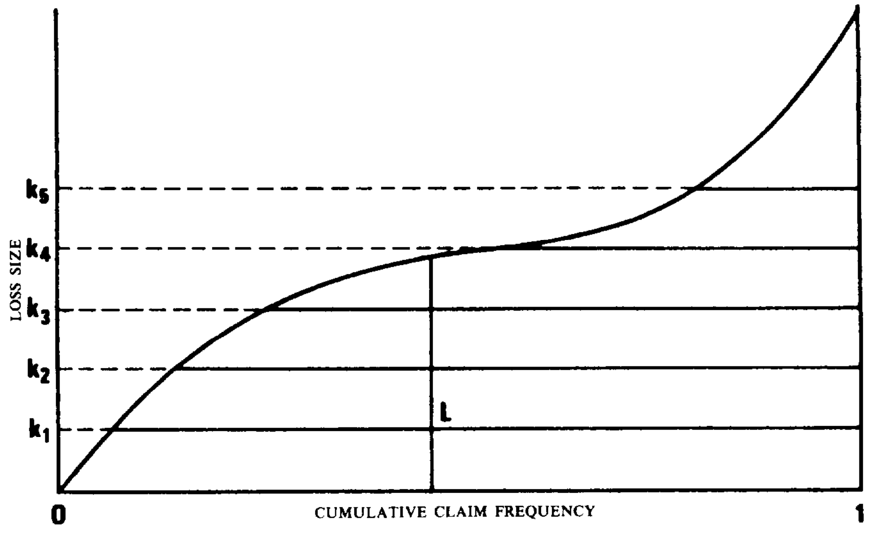


FIGURE 7
CONSISTENCY OF INCREASED LIMIT FACTOR



are of different widths, this property holds between the layers for the expected payment per unit coverage. Hence, the increased limits factor must increase at a decreasing rate as the limit increases. This is the consistency test. Actually, Figure 7 also shows that this is a common sense argument; a loss must have penetrated a lower layer before it reaches an upper layer.

Excess of Loss Coverage

An excess of loss contract generally covers losses in excess of a retention R, subject to a maximum limit L. The payment under such a contract may be expressed as a function of the loss X:

$$h\,(X; R, L) = \begin{cases} 0, & 0 < X \leq R \\ X - R, & R < X \leq S \\ L, & S < X, \end{cases} \tag{2.8}$$

where

$$S = R + L. \tag{2.9}$$

The situation may be described by means of the graph in Figure 8. For a loss such as represented by the line L_1 or L_2, the payment is represented by that portion of the line which falls inside the shaded region $BGEC$. The expected payment per ground-up claim under such contract has been derived in the literature by the size method, and can be expressed in many different forms; the following are given in Miccolis.

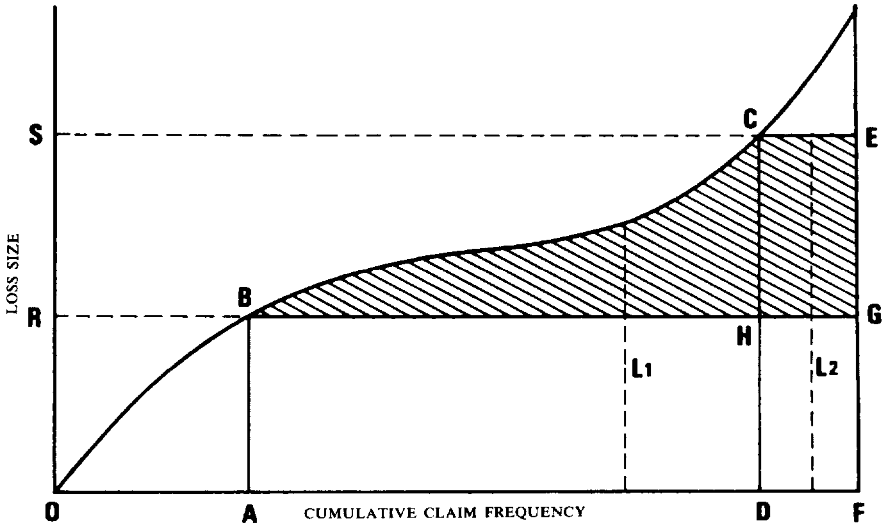
$$E\,\{h\,(X; R, L)\} = \int_R^S (x - R)\,dF\,(x) + LG\,(S) \tag{2.10}$$

$$= \int_R^S xdF\,(x) - R\,[F\,(S) - F\,(R)] + LG\,(S) \tag{2.11}$$

$$= \int_R^S xdF\,(x) + SG\,(S) - RG\,(R). \tag{2.12}$$

Figure 8 gives a simple graphical explanation of these integration results. They can be expressed in terms of the areas of the various regions shown in the graph, respectively as follows.

FIGURE 8
LOSSES WITH RETENTION AND LIMIT



$$E\{h(X; R, L)\} = BHC + HGEC \tag{2.13}$$

$$= ADCB - ADHB + HGEC \tag{2.14}$$

$$= ADCB + DFEC - AFGB. \tag{2.15}$$

Each of these is equal to the shaded area in the graph.

It is, of course, much easier to express the expected payment of such an excess of loss contract by the layer method:

$$E\{h(X; R, L)\} = \int_R^S G(x)\,dx. \tag{2.16}$$

The result is plain from Figure 8; it can also be derived from the integral expressions given above via integration by parts.

Relationships in the mathematics of excess of loss coverages could take on very complicated algebraic form, sometimes concealing the simplicity of the underlying idea. For example, Patrik [6] gives an expression for the expected loss excess of R subject to an upper limit of L in terms of $E\{X\} - R$ and other quantities. The average excess loss per ground-up claim is given by

$$E\{X\} - R + \Pr\{X \le R\} \cdot (R - E\{X|X \le R\}) - \Pr\{X \ge R+L\} \cdot [E\{X|X \ge R+L\} - (R+L)]. \quad (2.17)$$

This can be demonstrated by the graph in Figure 9 where A, B, C, and D represent areas of the respective regions. The above relation says simply that

$$B = (A + B + C) - (A + D) + D - C, \quad (2.18)$$

because

$$B = \text{expected excess loss} \quad (2.19)$$

$$A + B + C = E\{X\}, \text{ i.e. expected loss} \quad (2.20)$$

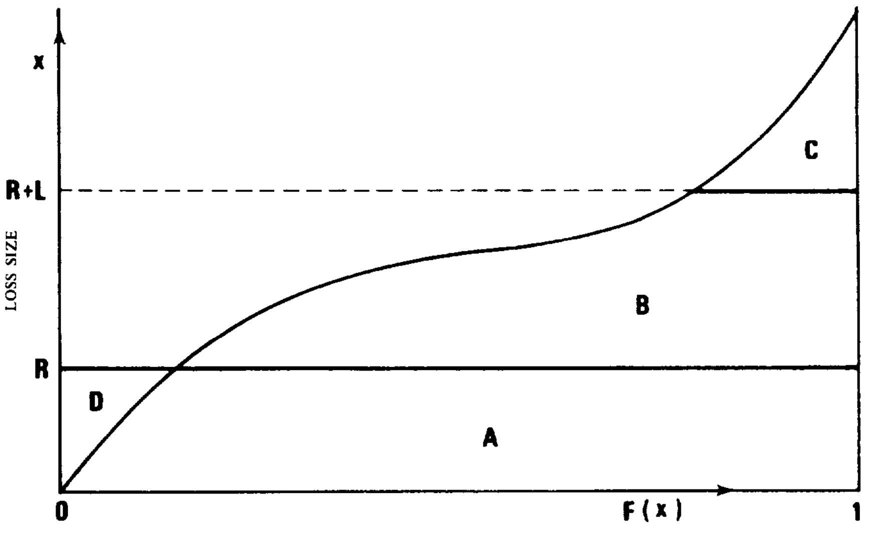
$$A + D = R \quad (2.21)$$

$$D = \Pr\{X \le R\} \cdot (R - E\{X|X \le R\}) \quad (2.22)$$

$$C = \Pr\{X \ge R+L\} \, [E\{X|X \ge R+L\} - (R+L)] \quad (2.23)$$

as is clear from the picture.

FIGURE 9
EXCESS OF LOSS COVERAGE



3. TREND

The effects of economic and social inflationary trends are to increase the size of losses. These effects act differently on the first dollar and the excess of loss coverages. Suppose the effect of inflation is, after a period of time, to change a loss of size x to a loss of size x', such that

$$x' = \alpha(x). \tag{3.1}$$

Assume that $\alpha(x)$ is a monotonic function, and let $F_1(x')$ be the cdf of x', i.e., the cdf after inflation. Then

$$F_1(x') = F(x), \tag{3.2}$$

and

$$F_1(\alpha(x)) = F(x). \tag{3.3}$$

The effect of inflation is demonstrated in Figure 10, where the lower curve represents the cdf before inflation, and the upper curve represents the cdf after inflation. The graph shows that a loss AB of size x becomes a loss AC of size x'. When, starting from the cdf curve $F(x)$, each size of loss, as represented by the vertical distance from the horizontal axis to the curve $F(x)$, is extended according to the function $x' = \alpha(x)$, we obtain the cdf curve after inflation. A simple case of inflation is one in which the loss is increased by a uniform multiplicative factor a, so that

$$x' = ax. \tag{3.4}$$

In this case, the cdf curve after inflation, $F_1(x')$, is obtained by extending each loss before inflation by a constant factor a.

It is well known that an excess of loss coverage is more seriously affected by inflation (assuming, for example, a uniform rate for all loss sizes); see, for example, Ferguson [2]. Figure 11 gives a dramatic demonstration of the leveraged effect of inflation on the excess of loss coverage. Let the rate of inflation be uniform for all sizes of loss, and the cdf curve after inflation be constructed from the curve before inflation as described above. The additional amount of loss resulting from inflation is shown in Figure 11 as the more heavily shaded region. If the retention R remains fixed, the expected excess loss payment is increased proportionally much more than indicated by the general rate of inflation.

FIGURE 10
EFFECT OF INFLATION

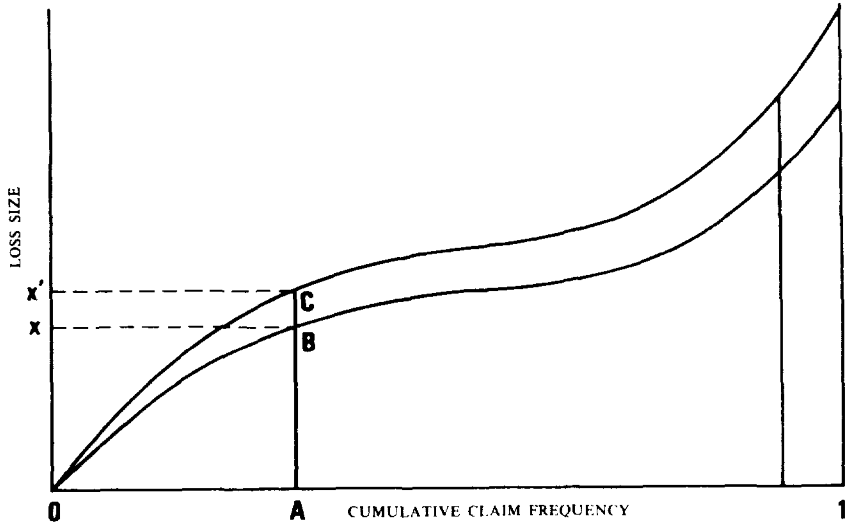
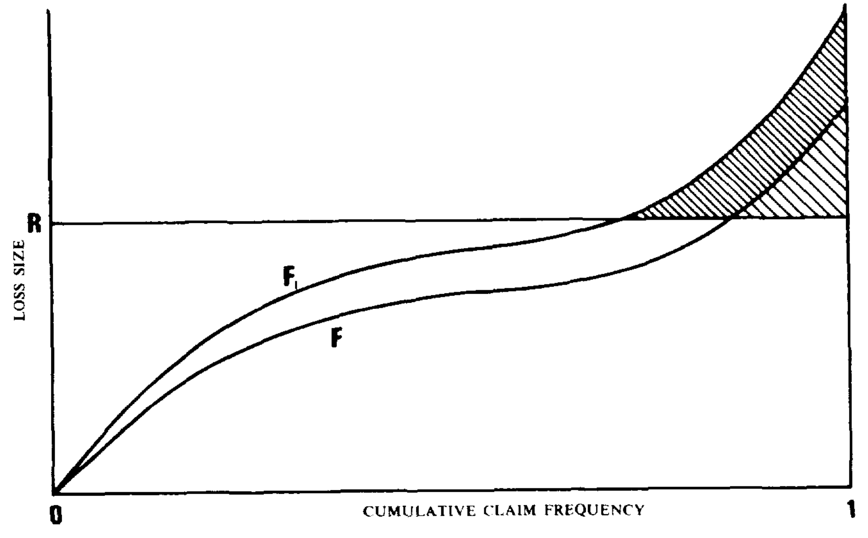


FIGURE 11
EFFECT OF INFLATION ON EXCESS LOSSES



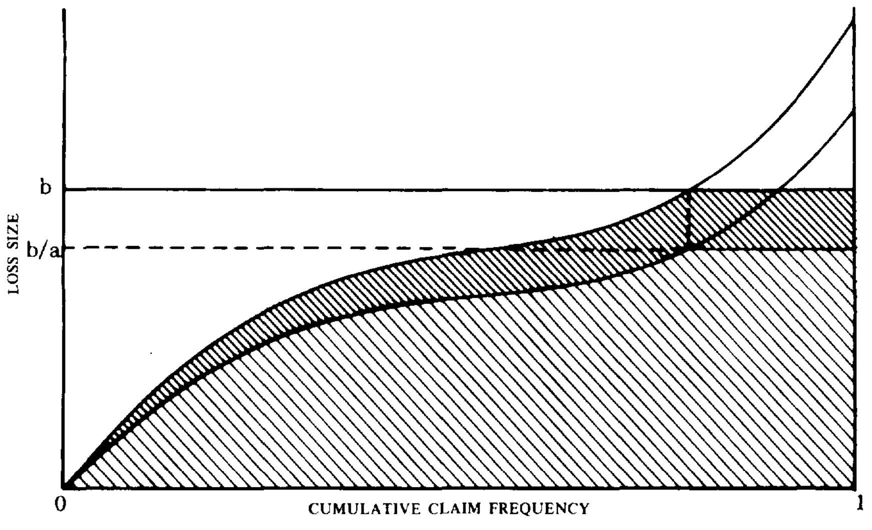
Since the total increase due to inflation is divided between the basic limits loss and the excess loss, the basic limits loss is expected to incur an inflationary increase at a lower rate than the total limits rate. This topic has been treated in Finger [3]. Figure 12 gives a graphical demonstration of this effect and also shows the following algebraic result (see, for example, Miccolis [5]):

$$E\{g(X';b)\} = a\,E\{g(X;b/a)\}. \tag{3.5}$$

The picture says that the new expected basic limits loss, represented by the total shaded area, is equal to the old expected loss up to the limit b/a, represented by the more lightly shaded area, extended by a factor a. Any vertical line through the shaded region in Figure 12 would have its intercept in the more heavily shaded region equal to a times its intercept in the more lightly shaded region.

FIGURE 12

EFFECT OF INFLATION ON BASIC LIMIT LOSSES



The study of the effect of inflation on excess of loss coverages can lead to rather complicated algebraic expressions. For example, Ferguson [2] relates the pure premium of an excess of loss coverage with indexing to the pure premium of one without indexing, the difference being expressed as a discount on the coverage without indexing. In an excess of loss coverage with indexing, the retention increases with inflation. A moment's reflection shows that the discount can be determined by comparing the expected loss under one contract with that under another. Let $\bar{X}$ be the average excess loss trended and indexed, R be the retention, $a-1$ be the proportional increase due to inflationary trend, Δ' be excess cost (per claim) on claims that exceed the retention as a result of inflation, and k be the multiplying factor which is equal to $G(R)$. Then Figure 13 shows that

$$E\{L_0\} = k\bar{X} + k(a-1)R + c\Delta', \tag{3.6}$$

$$E\{L_I\} = k\bar{X}, \tag{3.7}$$

where $E\{L_0\}$ is the expected excess loss per ground-up claim without indexing and $c = G(R) - G(aR)$ and $E\{L_I\}$ the expected excess loss per ground-up claim with indexing. Thus,

$$\Delta = 1 - \frac{E\{L_I\}\,E\{N\}}{E\{L_0\}\,E\{N\}} \tag{3.8}$$

$$= 1 - \frac{k\bar{X}}{k\bar{X} + k(a-1)R + c\Delta'}$$

$$= 1 - \frac{1}{1 + R(a-1)/\bar{X} + c\Delta'/k\bar{X}}\,. \tag{3.9}$$

or,

$$D = 1 - \frac{1}{1 + R(a-1)/\bar{X}} \tag{3.10}$$

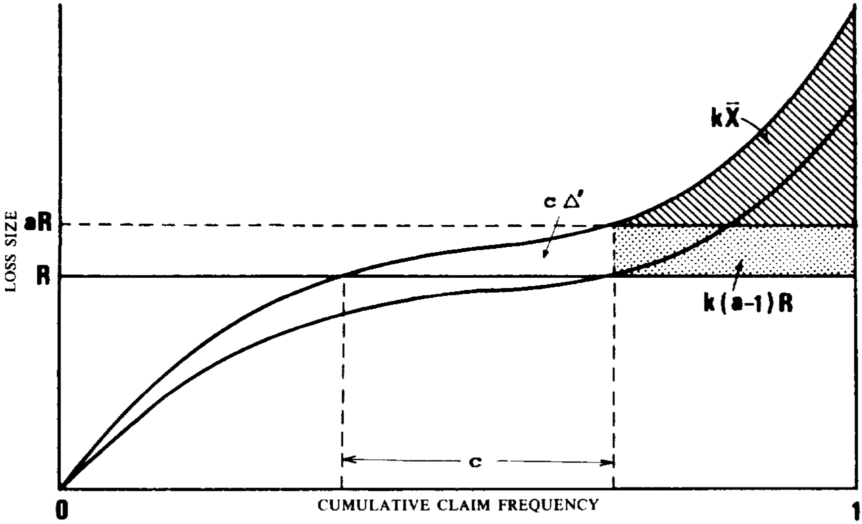
as proposed by Ferguson, neglecting the relatively small term involving Δ'.

4. RETROSPECTIVE RATING

The Excess Pure Premium Ratio

We first consider the mathematics of the excess pure premium ratio, commonly denoted by $\phi(r)$. This is defined to be a risk's average amount of loss in excess of r times its expected loss, divided by the expected loss. It is also known as the Table M charge, while the Table M savings,

FIGURE 13
INDEXING EXCESS OF LOSS COVERAGE



$\Psi(r)$, at the entry r (meaning r times the expected loss) is defined as the expected amount by which the risk's actual loss falls short of r times the expected loss, divided by the expected loss. More precisely, let

A = actual loss of the risk;
E = $E\{A\}$, the expected loss;
Y = A/E, actual loss in units of expected loss; and
$F(Y)$ = the cumulative distribution function of Y.

Then

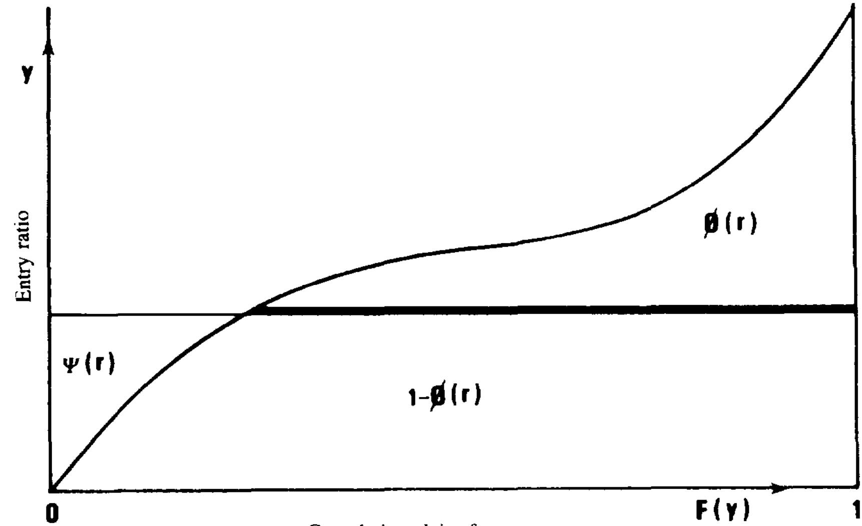
$$\phi(r) = \int_r^\infty (y - r)dF(y) \tag{4.1}$$

and

$$\psi(r) = \int_0^r (r - y)dF(y). \tag{4.2}$$

These functions are illustrated in Figure 14, where the cdf $F(y)$ is graphed against the entry ratio y. The functions $\phi(r)$ and $\psi(r)$ are represented by the areas indicated in the graph. A number of mathematical properties are now clearly demonstrated.

FIGURE 14
FUNCTIONS IN RETROSPECTIVE RATING



(1) By definition, the bounded area below the $F(y)$ curve is equal to 1. Hence $\phi(0) = 1$.
(2) $\phi(r)$ is a decreasing function of r, and $\phi(r) \rightarrow 0$ as $r \rightarrow \infty$.
(3) $\psi(r)$ is an increasing function of r; its value is unbounded as $r \rightarrow \infty$.
(4) Consider the small strip at $y = r$ in the graph. This shows that an increment dr from r will yield a decrease $G(r)dr$ in $\phi(r)$. Hence

$$\phi'(r) = (d/dr)\, \phi(r) = -G(r). \tag{4.3}$$

A second differentiation yields

$$\phi''(r) = f(r), \tag{4.4}$$

where $f(r)$ is the density function of the entry ratio, a result well known in the literature (Valerius [11]). Similarly, we may deduce from Figure 14 that

$$\psi'(r) = (d/dr)\, \psi(r) = F(r) \tag{4.5}$$

and

$$\psi''(r) = f(r). \tag{4.6}$$

(5) Consider the area of the rectangle on the interval from 0 to r in Figure 14. This gives the relation

$$r = [1 - \phi(r)] + \psi(r), \tag{4.7}$$

or

$$\psi(r) = \phi(r) + r - 1; \tag{4.8}$$

this is a fundamental relation connecting $\psi(r)$ and $\phi(r)$.

A result more general than (5) can also be obtained quite easily from Figure 15. Let

$$L = \begin{cases} r_1E & \text{if } A \leq r_1E \\ A & \text{if } r_1E < A \leq r_2E \\ r_2E & \text{if } r_2E < A. \end{cases} \tag{4.9}$$

Then the cdf of L/E can be represented by the solid line in Figure 15. The shaded area represents the quantity $E\{L\}/E$ and we have

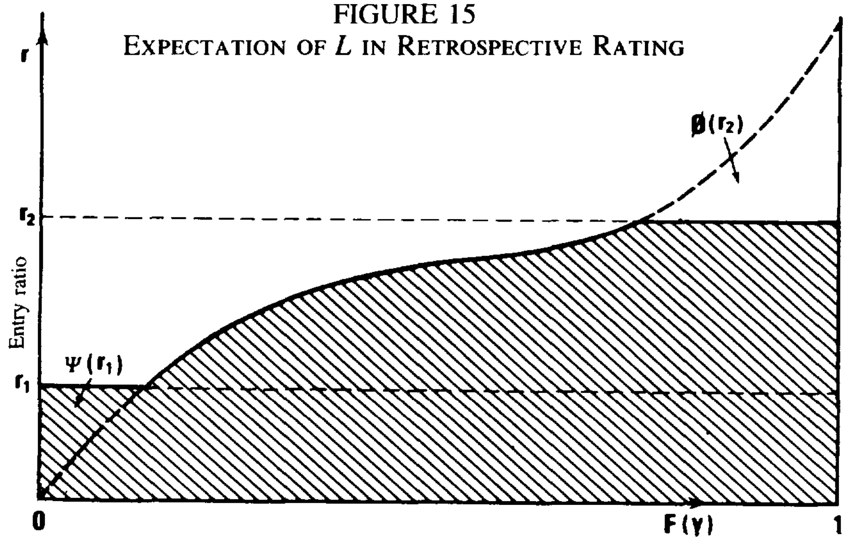
$$E\{L\}/E - \psi(r_1) + \phi(r_2) = 1, \tag{4.10}$$

or

$$E\{L\}/E = 1 + \psi(r_1) - \phi(r_2). \tag{4.11}$$

See Skurnick [9].

FIGURE 15
EXPECTATION OF L IN RETROSPECTIVE RATING



Retrospective Rating

In the Workers' Compensation Retrospective Rating Plan, the retrospective premium R is given by

$$R = b + CA, \tag{4.12}$$

subject to a maximum premium G and a minimum premium H, where b is the basic premium and C is the loss conversion factor (LCF), and where b is alternatively represented by

$$b = BP, \tag{4.13}$$

with P as the standard premium (before any applicable expense gradation) and B as the basic premium ratio. Let L_G be actual loss that will produce the maximum premium:

$$G = b + CL_G \tag{4.14}$$

and let

$$r_G = L_G/E. \tag{4.15}$$

Similarly, define L_H to be

$$H = b + CL_H, \tag{4.16}$$

$$r_H = L_H/E. \tag{4.17}$$

Further, let

$$L = \begin{cases} L_H & \text{if } A \leq L_H \\ A & \text{if } L_H < A \leq L_G \\ L_G & \text{if } L_G < A. \end{cases} \tag{4.18}$$

Then the retrospective premium can be represented by

$$R = b + CL. \tag{4.19}$$

For ease of exposition, we ignore the tax factor. If we identify r_H and r_G with r_1 and r_2, respectively, then Figure 16 shows the quantity $E\{L\}/E$ as the area of the shaded region $OFDCBA$. It then follows that

$$E\{L\} = E - \phi(r_G)E + \psi(r_H)E \tag{4.20}$$

$$= E - I, \tag{4.21}$$

where

$$I = E[\phi(r_G) - \psi(r_H)] \tag{4.22}$$

is called the net insurance charge of Table M. If the plan is to be balanced, the expected retrospective premium must be equal to the sum of the total expenses, e, and the expected loss, E:

$$E\{R\} = e + E. \tag{4.23}$$

On the other hand, it also follows from the above that

$$E\{R\} = b + C(E - I). \tag{4.24}$$

Equating these two quantities we obtain the basic premium in terms of the expense, expected loss, and the net insurance charge:

$$b + C\{E - I) = e + E \tag{4.25}$$

or

$$b = e - (C - 1)E + CI. \tag{4.26}$$

A formula relating the charge difference to the minimum premium, expected loss and expense provision has been used to facilitate the determination of retrospective rating values from specified maximum and minimum premiums. This formula can be derived with the help of Figure 16.

Consider the equation

$$R = b + CL \tag{4.27}$$

Taking the expectation and representing the expectation $E\{L\}/E$ by the shaded area of Figure 16 we have

$$e + E = b + CE[OFDCBA]. \tag{4.28}$$

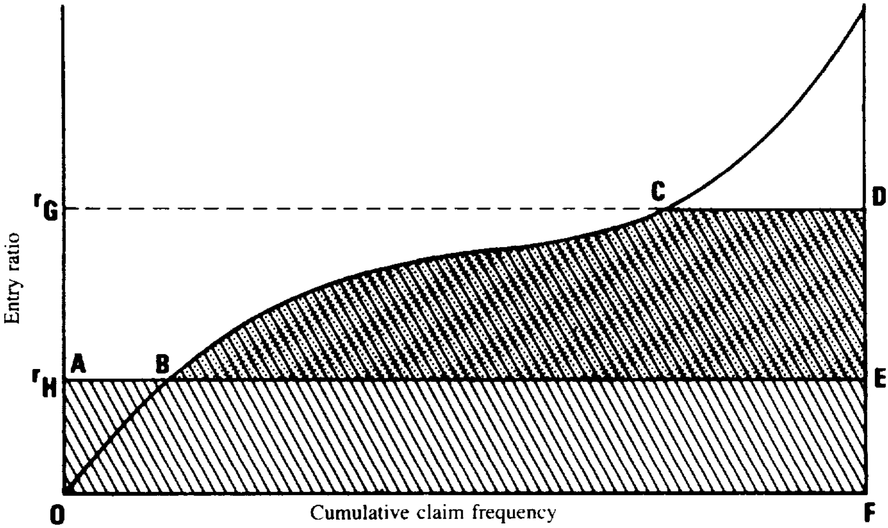
On the other hand, we have for the minimum premium H:

$$H = b + CEr_H \tag{4.29}$$
$$= b + CE\ [OFEA]. \tag{4.30}$$

Taking the difference on both sides of the two equations above we have

$$(e + E) - H = CE\ [BEDC] \tag{4.31}$$
$$= CE\ [\phi(r_H) - \phi(r_G)]. \tag{4.32}$$

FIGURE 16
RETROSPECTIVE RATING PREMIUM



This formula, together with the formula

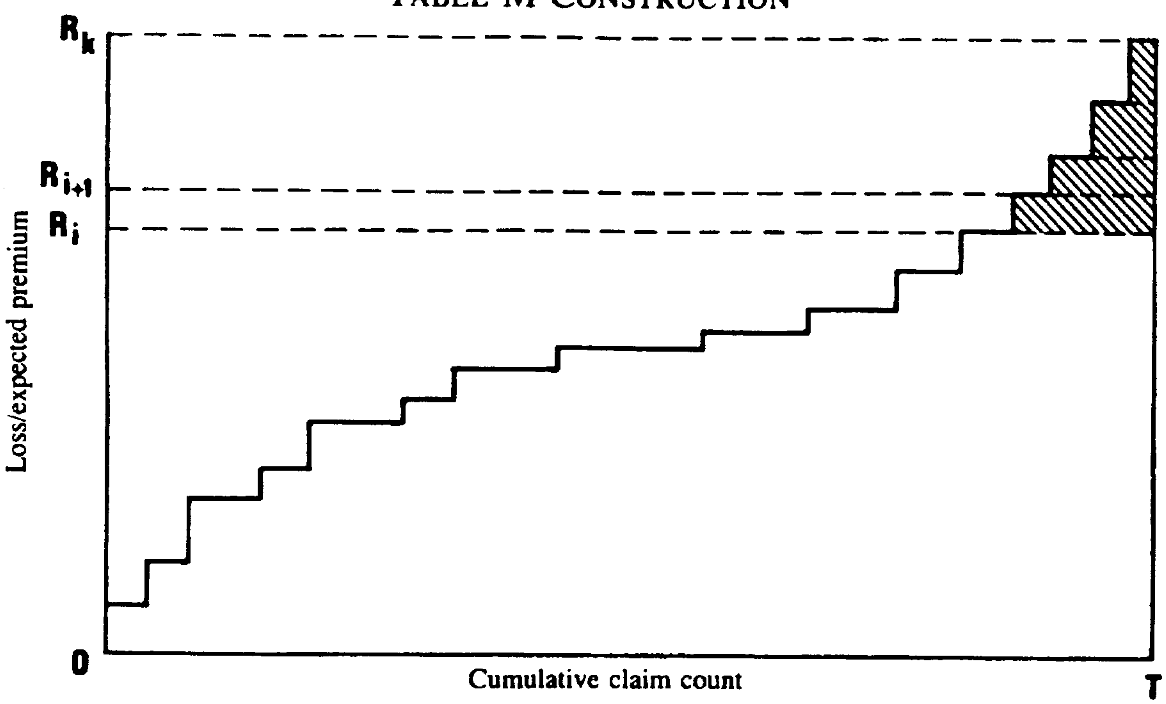
$$G - H = CE(r_G - r_H), \tag{4.33}$$

which is much easier to derive, can be used to determine the rating values given the maximum and minimum premiums. One may interpret the difference in charge, $\phi(r_H) - \phi(r_G)$, as indicated by the dotted area in Figure 16, to be the difference between the expected retrospective premium and the minimum premium, apart from a conversion factor CE.

Construction of Table M

A Table M has been constructed by Simon [8]; see also Skurnick [9]. The algebra involved in the construction procedure appears to be rather complicated. Actually, the idea is very simple when this is expressed in a graph. Figure 17 shows a cumulative frequency curve constructed from observed data on risks within a premium group. Let the loss ratios be arranged in ascending order: $R_1, R_2, \ldots, R_k$, with R_i occurring N_i times. Also let the total number of claims be $T = N_1 + \ldots + N_k$. The cumulative frequency up to R_i, i.e. $T_i = N_1 + \ldots + N_i$ is plotted against R_i for each i so as to form a step function whose abscissa in the interval (R_i, R_{i+1}) is the cumulative frequency T_i, as shown in Figure 17. We may think of this graph as a re-scaled version of the cdf curve plotted against the entry ratio. It now appears quite clear that the value of ϕ for the entry ratio corresponding to R_i is simply the shaded area in Figure 17 divided by the total enclosed area below the cumulative frequency curve. The entry ratio corresponding to R_i is simply R_i divided by the average loss ratio $\Sigma N_i R_i / T$.

FIGURE 17
TABLE M CONSTRUCTION



A convenient procedure to construct a Table M is to sum the horizontal strips downward, cumulatively, starting from the strip corresponding to (R_{k-1}, R_k), down to the strip corresponding to $(0, R_1)$. It is convenient also to sum the frequencies downward, cumulatively, because the cumulative sum of such frequencies down to and including N_{i+1} is the length of the strip corresponding to the interval (R_i, R_{i+1}). Thus let

$$S_{1,i} = \sum_{j=i+1}^{k} N_j \tag{4.34}$$

which is represented by the length of the strip on (R_i, R_{i+1}), and

$$S_{2,i} = S_{2,i+1} + S_{1,i} (R_{i+1} - R_i) \tag{4.35}$$

which describes the fact that the sum of the strips above R_i is obtained by adding the strip on (R_i, R_{i+1}) to the sum of the strips above R_{i+1}. The value of ϕ at the entry ratio corresponding to R_i is then $S_{2,i}/S_{2,0}$, with $S_{2,0}$ equal to the total area of all the strips. The entry ratio corresponding to R_i is obtained by normalization:

$$r_i = R_i \Big/ \left(\frac{S_{2,0}}{S_{1,0}}\right) . \tag{4.36}$$

We may think of R_i as loss expressed in an arbitrary unit and the denominator as the expected loss in this unit. The procedure is described in algebraic form by Skurnick. It is easy to see that this is a layer approach.

Table L

A retrospective rating plan may provide for a per accident limit on losses. The table of charges which incorporates this per accident limitation is called Table L, which has been described by Skurnick [9]. Let A be the actual unlimited loss, as before, A^* be the actual limited loss, and $F^*(.)$ be the cdf of $Y^* = A^*/E$. Then the Table L charge is defined as (Skurnick)

$$\phi^*(r) = \int_r^\infty (y - r)dF^*(y) + k, \tag{4.37}$$

where k is the loss elimination ratio

$$k = [E - A^*]/E \tag{4.38}$$

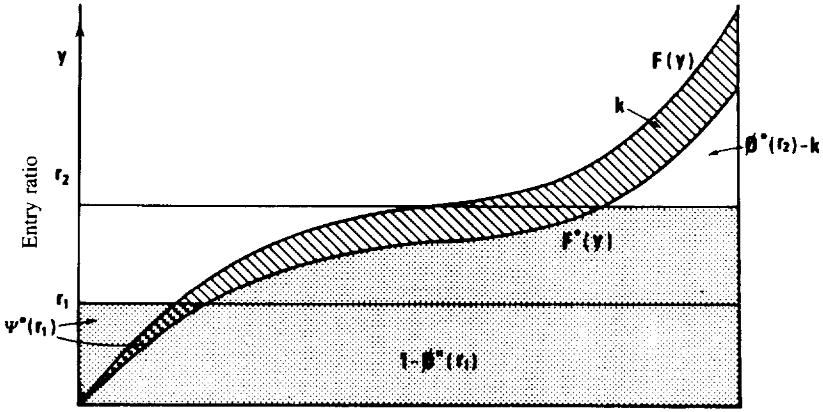
Further, the Table L savings are defined as

$$\psi^* = \int_0^r (r - y)dF^*(y). \tag{4.39}$$

In Figure 18 the curves for $F(y)$ and $F^*(y)$ are plotted against the entry ratio $r = A/E$. $F(y)$ is necessarily situated above $F^*(y)$, and by the definition of r, the enclosed area below the $F(y)$ curve is equal to 1, while the enclosed area below the $F^*(y)$ curve is $1 - k$. The area of the shaded belt is equal to the loss elimination ratio k. Many of the properties of the Table L charges, as presented by Skurnick [9], can be easily obtained from the graph. For example, consider the limited loss

$$L^* = \begin{cases} r_1E & \text{if } A^* \leq r_1E \\ A^* & \text{if } r_1E < A^* \leq r_2E \\ r_2E & \text{if } r_2E < A^*. \end{cases} \tag{4.40}$$

FIGURE 18
TABLE L FUNCTIONS



Then $E\{L^*\}/E$ is represented by the dotted area in Figure 18. We deduce that

$$E\{L^*\}/E - \psi^*(r_1) + [\phi^*(r_2) - k] = 1 - k \tag{4.41}$$

and hence

$$E\{L^*\}/E = 1 + \psi^*(r_1) - \phi^*(r_2), \tag{4.42}$$

as in Skurnick. As another example, identify r_1 and r_2, respectively, with r_H and r_G as defined before. Also let

$$R^* = b^* + CL^* \tag{4.43}$$

be the retrospective premium with per accident limitation. Then, combining the equation

$$E\{R^*\} = e + E = b^* + CEr_H + CE[\phi^*(r_H) - \phi^*(r_G)], \tag{4.44}$$

which follows from the fact that the expected retrospective premium is b^* plus the dotted area (converted), with the equation

$$H = b^* + CEr_H, \tag{4.45}$$

we have the Table L version of a familiar formula

$$e + E - H = CE\,[\phi^*(r_H) - \phi^*(r_G)], \tag{4.46}$$

the last factor on the right being represented by the dotted area between $r_1 = r_H$ and $r_2 = r_G$ in Figure 18. As a final example of the use of Figure 18, one may consider the construction of Table L. This can be done in a manner similar to the construction of Table M, except that the cumulative frequency function of the limited loss is used, and the final result has to be adjusted for the loss elimination factor k.

Asymptotic Behavior

As the premium size becomes large, the limiting form of the charge takes on a simple function. The graphs in Figure 19 help us to understand the asymptotic behavior. Consider the case with no per loss limitation.

FIGURE 19
LIMITING CASE IN RETROSPECTIVE RATING

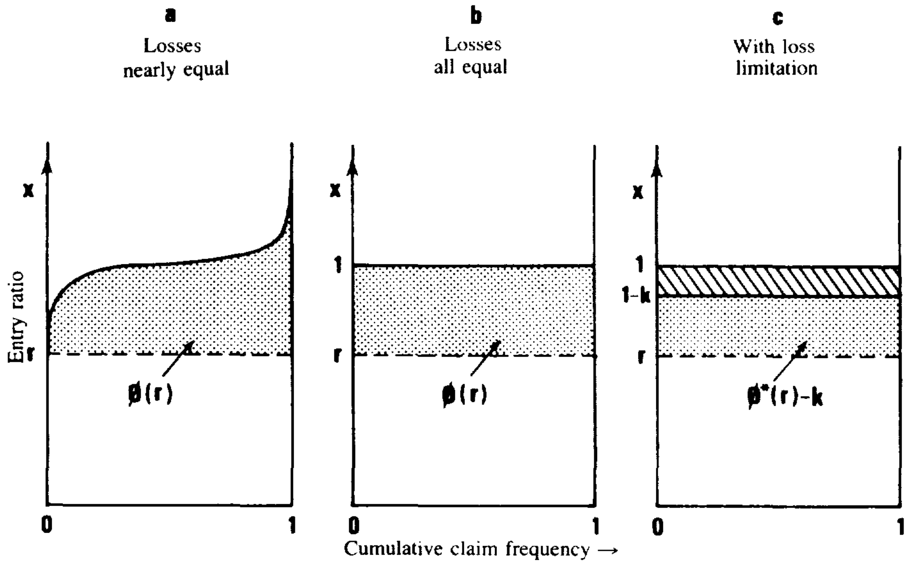


Figure 19(a) shows a cdf curve for losses which are nearly equal; here the $\phi(r)$ region almost forms a rectangle. When all losses are equal, the cdf $F(x)$ is a step function with a single jump at $x = 1$, as shown in Figure 19(b). The Table M charge, $\phi(r)$, at the entry point r is represented by the area of the rectangle between r and 1. Hence

$$\phi(r) = \begin{cases} 1 - r & r \leq 1 \\ 0 & 1 < r. \end{cases} \tag{4.47}$$

The limiting case with per loss limitation is shown in Figure 19(c). Here the cdf $F^*(x)$ is shown as the horizontal line $x = 1 - k$, where it has its single jump. The Table L charge, $\phi^*(r)$, is the area of the rectangle between r and $1 - k$, plus the loss elimination ratio k. Thus

$$\phi^*(r) = \begin{cases} 1 - r & r < 1 - k \\ k & 1 - k \leq r. \end{cases} \tag{4.48}$$

Other Applications

There are other interesting mathematical relationships in the mathematics of retrospective rating, and many such intricate relationships are presented in Carlson [1]. It is a great burden to follow the algebra of the many complicated relationships presented there. Most of these, however, become much clearer if we make use of the graphical approach adopted here. Rather than go through the numerous equations and formulas in Carlson, we present a particular example to illustrate the power of our graphical method. Let us pick, almost at random, equation (15a) in Carlson, which can be explained as follows. Let the minimum premium be greater than the basic premium, and the maximum premium be equal to the standard premium:

$$H > B, \ G = P. \tag{4.49}$$

Then, in Carlson's notation,

$$P - Rv = C(P's - H's) \tag{4.50}$$

$$= C(P' - H') - C(H'p - P'p). \quad 4.51)$$

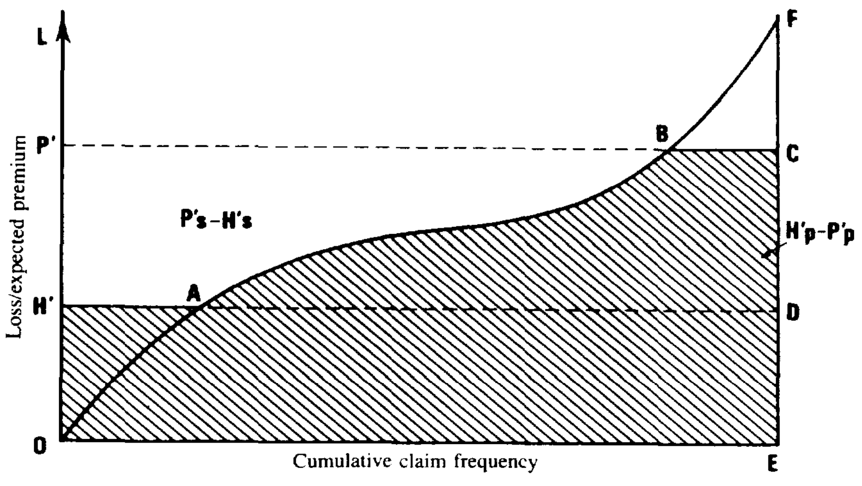
These equations follow immediately from Figure 20 with the following interpretation of Carlson's notations:

$$P \;= b + CP' \tag{4.52}$$

$$Rv = \text{expected retrospective premium} \tag{4.53}$$

$$= b + C[OECBAH'] \tag{4.54}$$

FIGURE 20
RELATIONSHIPS IN RETROSPECTIVE RATING



$$P's = OBP' \tag{4.55}$$

$$H's = OAH' \tag{4.56}$$

$$H'p = ADF \tag{4.57}$$

$$P'p = BCF. \tag{4.58}$$

5. CONCLUSION

This paper presents a graphical approach to the mathematics of excess of loss coverages and related topics. The graphs serve to simplify and clarify much of the complicated algebra which has hitherto been the sole vehicle to express the mathematical ideas involved. We hope this will become a useful addition to the actuarial tool box of the student and the practicing casualty actuary alike. This technique has been used in explaining the principles of coinsurance and its many properties (Lee [4]). Philbrick [7] uses the same idea to describe size of loss distributions.

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