Maths 190 (Math'l Models in Finance)

SOLUTIONS TO PRACTICE PROBLEMS
Assignment No. 2

You will learn and benefit more if you attempt solving these problems first before looking at their solutions.

BINOMIAL OPTION PRICING PROBLEMS

Problem 12.2

Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

Answer: In the no-arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on the portfolio equal to the risk-free interest rate, we are able to value the option. When we use risk-neutral valuation, we first choose probabilities for the branches of the tree so that the expected return on the stock equals the risk-free interest rate. We then value the option by calculating its expected payoff and discounting this expected payoff at the risk-free interest rate.

Problem 12.3

What is meant by the delta of a stock option?

Answer: The delta of a stock option measures the sensitivity of the option price to the price of the stock when small changes are considered. Specifically, it is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

Problem 12.5

A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100?

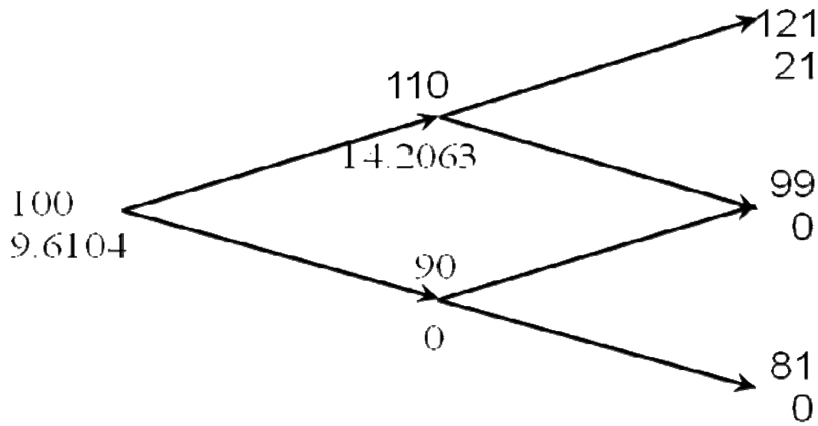
Answer: In this case $u = 1.10$, $d = 0.90$, $\Delta t = 0.5$, and $r = 0.08$, so that

$$p = \frac{e^{0.08\times 0.5} - 0.90}{1.10 - 0.90} = 0.7041$$

The tree for stock price movements is shown below. We can work back from the end of the tree to the beginning, as indicated in the diagram, to give the value of the option as \$9.61. The option value can also be calculated directly using the risk-neutral valuation formula:

$$[0.7041^2 \times 21 + 2 \times 0.7041 \times 0.2959 \times 0 + 0.2959^2 \times 0]e^{-2\times 0.08\times 0.5} = 9.61$$

or $9.61.



Binomial tree for problem 12.5

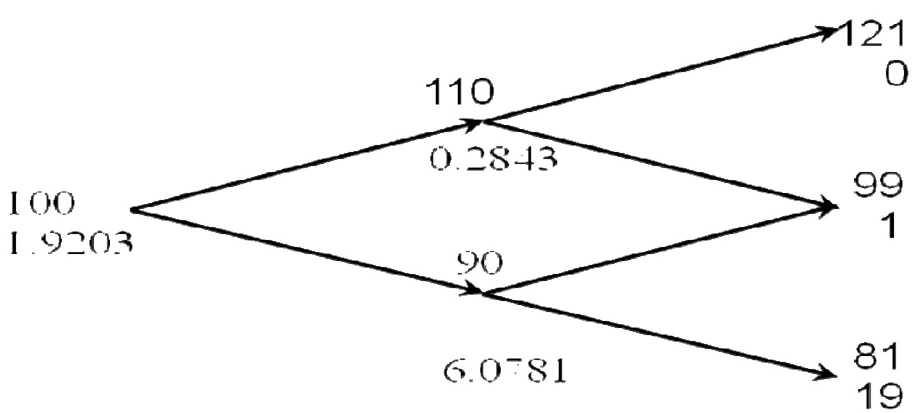
Problem 12.6

For the situation considered in Problem 12.5, what is the value of a one-year European put option with a strike price of $100? Verify that the European call and European put prices satisfy put–call parity.

Answer: The figure below shows how we can value the put option using the same tree as in Problem 12.5. The value of the option is $1.92. The option value can also be calculated directly from the risk-neutral valuation formula:

$$e^{-2\times 0.08\times 0.5}[0.7041^2 \times 0 + 2 \times 0.7041 \times 0.2959 \times 1 + 0.2959^2 \times 19] = 1.92$$

or $1.92. The stock price plus the put price is $100 + 1.92 = \$101.92$. The present value of the strike price plus the call price is $100e^{-0.08\times 1} + 9.61 = \$101.92$. These are the same, verifying that put–call parity holds.



Binomial tree for problem 12.6

Problem 12.8

Consider the situation in which stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

Answer: The riskless portfolio consists of a short position in the option and a long position in Δ shares. Because Δ changes during the life of the option, this riskless portfolio must also change.

Problem 12.10

A stock price is currently $80. It is known that at the end of four months it will be either $75 or $85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a four-month European put option with a strike price of $80? Use no-arbitrage arguments.

Answer: At the end of four months the value of the option will be either $5 (if the stock price is $75) or $0 (if the stock price is $85). Consider a portfolio consisting of:

$$-\Delta \quad : \quad \text{shares}$$

$$+1 \quad : \quad \text{option}$$

(Note: The delta, Δ of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either -85Δ or $-75\Delta+5$ in four months. If

$$-85\Delta = -75\Delta + 5$$

i.e.,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 42.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.5\times 80 + f$$

where f is the value of the option. Since the portfolio is riskless

$$(0.5\times 80 + f)e^{0.05\times 4/12} = 42.5$$

i.e.,

$$f = 1.80$$

The value of the option is therefore $1.80.

This can also be calculated directly using the risk-neutral valuation principle. With $u = 1.0625$ and $d = 0.9375$, the risk-neutral probability p (or q in our notation in class), we have

$$p = \frac{e^{0.05\times 4/12} - 0.9375}{1.0625 - 0.9375} = 0.6345$$

$1 - p = 0.3655$ and

$$f = e^{-0.05\times 4/12}\times 0.3655\times 5 = 1.80$$

Problem 12.11

A stock price is currently \$40. It is known that at the end of three months it will be either \$45 or \$35. The risk-free rate of interest with quarterly compounding is 8% per annum. Calculate the value of a three-month European put option on the stock with an exercise price of \$40. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

Answer: At the end of three months the value of the option is either \$5 (if the stock price is \$35) or \$0 (if the stock price is \$45).

Consider a portfolio consisting of:

$$-\Delta \quad : \quad \text{shares}$$

$$+1 \quad : \quad \text{option}$$

(Note: The delta, Δ, of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either $-35\Delta+5$ or -45Δ. If:

$$-35\Delta+5=-45\Delta$$

i.e.,

$$\Delta=-0.5$$

the value of the portfolio is certain to be 22.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is

$$-40\Delta+f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(40\times 0.5+f)\times 1.02=22.5$$

Hence

$$f=2.06$$

i.e., the value of the option is \$2.06.

This can also be calculated using risk-neutral valuation. Suppose that p (q in our notation in the class) is the probability of an upward stock price movement in a risk-neutral world. We must have

$$45p+35(1-p)=40\times 1.02$$

i.e.,

$$10p=5.8$$

or:

$$p=0.58$$

The expected value of the option in a risk-neutral world is:

$$0\times 0.58+5\times 0.42=2.10$$

This has a present value of

$$\frac{2.10}{1.02}=2.06$$

This is consistent with the no-arbitrage answer.

Problem 12.14
A stock price is currently \$25. It is known that at the end of two months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of two months. What is the value of a derivative that pays off S_T^2 at this time?

Answer: At the end of two months the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

$$+\Delta \quad : \quad \text{shares}$$

$$-1 \quad : \quad \text{derivative}$$

The value of the portfolio is either $27\Delta - 729$ or $23\Delta - 529$ in two months. If

$$27\Delta - 729 = 23\Delta - 529$$

i.e.,

$$\Delta = 50$$

the value of the portfolio is certain to be 621. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$50 \times 25 - f$$

where f is the value of the derivative. Since the portfolio must earn the risk-free rate of interest

$$(50 \times 25 - f)e^{0.10 \times 2/12} = 621$$

i.e.,

$$f = 639.3$$

The value of the option is therefore \$639.3.
This can also be calculated directly from the risk-neutral valuation principle. With $u = 1.08$, $d = 0.92$, the risk-neutral probability p (q in our notation in the class),

$$p = \frac{e^{0.10 \times 2/12} - 0.92}{1.08 - 0.92} = 0.6050$$

and

$$f = e^{-0.10 \times 2/12}(0.6050 \times 729 + 0.3950 \times 529) = 639.3$$

Problem 12.15 **[This is an important exercise. So, pay extra attention to the modification on the risk-neutral probability of an up/down movement.]**
Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is one month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

Answer: In this case

$$a = e^{(0.05 - 0.08) \times 1/12} = 0.9975$$

$$u = e^{0.12\sqrt{1/12}} = 1.0352$$

$$d = 1/u = 0.9660$$

$$p = \frac{0.9975 - 0.9660}{1.0352 - 0.9660} = 0.4553$$ (The p here is q in our notation in the class).

♣REQUIRED PROBLEM #1 [4 points]

A stock price is currently $25. It is known that at the end of 4 months it will be either $30 or $21. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a 4-month European call option with an exercise price of $24. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answer.

Solution given in a separate sheet.

♣REQUIRED PROBLEM #2 [5 points]

In a two-period binomial model with $r = 1\%$ per period, the current stock price is $100, and $u = 1.02$ and $d = 0.98$. Consider an option that expires after two periods, and pays the value of the squared stock price, $S(t)^2$, if the stock price $S(t)$ is higher than $100 when the option is exercised. Otherwise (when $S(t)$ is less than or equal to $100), the option pays zero. If the option under consideration is an American-type, find its price.

Solution given in a separate sheet.

BASIC ELEMENTS OF STOCHASTIC PROCESSES

♣REQUIRED PROBLEM #3 [7 points]

Consider a 2-step trinomial non-recombining lattice tree model. For each step, there are three possibilities for the stock price: an up movement (u), a down movement (d) or no movement (n).

(a) Write down the set or sample space, Ω, containing all possible outcomes for this 2-step trinomial tree. If we consider the collection of all subsets of Ω, how many subsets are there in this collection? [1 point]

(b) Suppose the respective probabilities of events $\{u\}$ and $\{d\}$ are 3/7 and 2/7. Define or construct the probability measure for each individual element $\omega \in \Omega$. [1.5 points]

(c) Write down the σ-algebra or σ-field $\{\Im_i\}$ keeping track the outcomes for each time step i = 0, 1, 2. [1.5 points]

(d) Consider the Ω given in (a). Under the trinomial asset pricing model suppose $S_0 = 50$, $d = 10/11$ and $u = 12/11$; clearly, $\{S_i\}$ is a stochastic process, i.e., S_i's are random variables for i = 0, 1, 2. Find $S_1(\varpi)$, i.e., what is the function $S_1(\varpi)$? [2 pts]

(e) Consider the interval $B=\left[2\pi^e-1,\ 2e^\pi+1\right]$. What is $S_1^{-1}(B)$? Recall that S_1 is a random variable and by definition it maps Ω into $\Re$. [1 pt]

Solution given in a separate sheet.

♣REQUIRED PROBLEM #4 [4 points]

Assume a single-period model with $r = 0.05$ and S(0) =100. After one period, the stock price can either go up to 101 or down to 99.

(a) Find a replicating portfolio for a put option with the pay off function $g(s) = (100-s)^+ = \max(100-s, 0)$. **Use a replicating portfolio consisting of α amount held in a savings account and Δ shares of holdings in stock.** [3 pts] [**Note:** *You will get a zero for this question if you construct a different replicating portfolio.*]

(b) Using your replicating portfolio in (a), what is the price of the put option in question? [1 pt]

Solution given in a separate sheet.

Additional Problem (Not for submission)

Consider a single-period CRR model with interest rate 0.05, $S(0)=10$, $u=1.2$ and $d=0.98$. Suppose you have written an option that pays the value of the square root of the absolute value of the difference between the stock price at maturity and \$10.00; that is, it pays $\sqrt{|S(1)-10|}$. How many shares of the stock should you buy to replicate this pay-off? What is the cost of the replicating portfolio?

Answer:

Let α and Δ be the respective holdings in the money market account and stock investments.

We need to have $\begin{array}{l} 1.05\alpha + 12\Delta = \sqrt{|12-10|} = 1.4142 \\ 1.05\alpha + 9.8\Delta = \sqrt{|9.8-10|} = 0.4472 \end{array}$

Solving this system we get $\alpha = -3.6765$ and $\Delta = 0.4395$.

Thus, we need to buy 0.4395 shares and the cost of the replicating portfolio is $\alpha + 10\Delta = 0.7189$.