Maths 190 (Math'l Models in Finance)
AY 2016-17
SOLUTIONS TO REQUIRED PROBLEMS
Assignment No. 2

♣REQUIRED PROBLEM #1 [4 points]
Additional Problem 1

A stock price is currently $25. It is known that at the end of 4 months it will be either $30 or $21. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a 4-month European call option with an exercise price of $24. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answer.

Answer:

At the end of four months the value of the option will be either $6 (if the stock price is $30) or $0 (if the stock price is $21). Consider a portfolio consisting of:

$$\begin{array}{ll} \Delta: & \text{shares} \\ -1: & \text{option} \end{array}$$

The value of the portfolio is either 30Δ-6 or 21Δ in four months. If

$$30\Delta-6 = 21\Delta$$

i.e.,

$$\Delta = 2/3 \quad \text{1 pt}$$

the value of the portfolio is certain to be 14. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$25\Delta - f = 25(2/3) - f$$

where f is the value of the option. Since the portfolio is riskless

$$[25(2/3) - f]e^{(0.12)(4/12)} = 14$$

i.e.,

$$\boxed{f = \$3.2156.} \quad \text{1 pt}$$

The value of the option is therefore $2.90.

This can also be calculated using the risk-neutral valuation approach, where u= 1.20 and d = 0.84 so that

$$q = \frac{e^{0.12(4/12)} - 0.84}{1.20 - 0.84} = 0.557807706 \quad \text{1 pt}$$

and 1- q = 0.442192294. Therefore,

$$\boxed{f = e^{-0.12(4/12)}(0.557807706)(6) = \$3.2156.} \quad \text{1 pt}$$

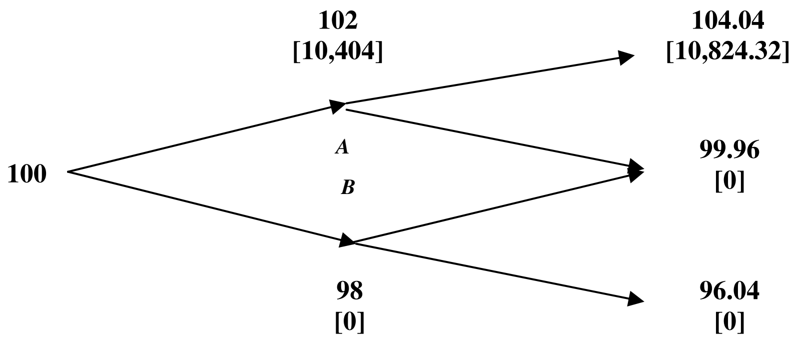
♣REQUIRED PROBLEM #2 [5 points]

Additional Problem 2

In a two-period binomial model with $r = 1\%$ per period, the current stock price is \$100, and $u = 1.02$ and $d = 0.98$. Consider an option that expires after two periods, and pays the value of the squared stock price, $S(t)^2$, if the stock price $S(t)$ is higher than \$100 when the option is exercised. Otherwise (when $S(t)$ is less than or equal to \$100), the option pays zero. If the option under consideration is an American-type, find its price.

Answer:

We present the corresponding tree for the stock price.



1 pt

In the brackets we record the payoff of the American option if we exercise it in the corresponding node. In the final nodes, the payoff is equal to the payoff of a European-style option.

In node B, the payoff and the value of the unexercised option is 0. 0.5 pts

In node A, if we exercise the option, the payoff is $102^2 = 10{,}404$. We have to compare this payoff with the value of the unexercised option. For that, we need to compute the risk-neutral probability q as 0.5 pts

$$q = \frac{(1+r)-d}{u-d} = \frac{1.01-0.98}{1.02-0.98} = 0.75 .$$

1 pt

The value of the unexercised option at that node is

$$\frac{1}{1.01}(0.75)(10{,}824.32) = 8{,}037.86 .$$

1 pt

Therefore, it is optimal to exercise early at this node (node A).

We compute the price of the option at the initial moment as

$$\frac{1}{1.01}(0.75)(10{,}404) = \$7{,}725.743$$

1 pt

♣REQUIRED PROBLEM #3 [7 points]

Consider a 2-step trinomial non-recombining lattice tree model. For each step, there are three possibilities for the stock price: an up movement (u), a down movement (d) or no movement (n).

(a) Write down the set or sample space, Ω, containing all possible outcomes for this 2-step trinomial tree. If we consider the collection of all subsets of Ω, how many subsets are there in this collection? [1 point]

(b) Suppose the respective probabilities of events $\{u\}$ and $\{d\}$ are 3/7 and 2/7. Define or construct the probability measure for each individual element $\omega \in \Omega$. [1.5 points]

(c) Write down the σ-algebra or σ-field $\{\Im_i\}$ keeping track the outcomes for each time step i = 0, 1, 2. [1.5 points]

(d) Consider the Ω given in (a). Under the trinomial asset pricing model suppose S_0 = 50, d = 10/11 and u = 12/11; clearly, $\{S_i\}$ is a stochastic process, i.e., S_i's are random variables for i = 0, 1, 2. Find $S_1(\varpi)$, i.e., what is the function $S_1(\varpi)$? [2 pts]

(e) Consider the interval $B=\left[2\pi^{e}-1,\ 2e^{\pi}+1\right]$. What is $S_1^{-1}(B)$? Recall that S_1 is a random variable and by definition it maps Ω into $\Re$. [1 pt]

Answer:

(a) All possible outcomes of tossing a coin three times can be described by the set
0.5 pts $\Omega = \{uu, un, ud, nu, nn, nd, du, dn, dd\}$. Since there are 9 elements in Ω, there would be 2^9 = 512 subsets of Ω contained in the collection.
0.5 pts

(b) For the individual elements of Ω, we have

$P(uu) = (3/7)^2 = 9/49$ $\qquad$ $P(nn) = (2/7)^2 = 4/49$

$P(un) = (3/7)(2/7) = 6/49$ $\qquad$ $P(nd) = (2/7)^2 = 4/49$

$P(ud) = (3/7)(2/7) = 6/49$ $\qquad$ $P(du) = (2/7)(3/7) = 6/49$

$P(nu) = (2/7)(3/7) = 6/69$ $\qquad$ $P(dn) = (2/7)^2 = 4/49$

$P(dd) = (2/7)^2 = 4/49$

Note that $\sum_{\omega\in\Omega} P\{\omega\} = 1$. 0.15 pts for each correct item

(c) $\Im_0 = \{\phi, \Omega\}$ signifying we do not know anything yet and we note that $\phi^c = \Omega \in \Im_0$

since $\Im$ is closed under set complementation. 0.25 pts

$$\Im_1 = \left\{ \begin{matrix} \phi, \Omega, \{uu, un, ud\}, \{nu, nn, nd\}, \{du, dn, dd\}, \text{and all sets which} \\ \text{can be built by taking unions and set complementation (i.e., intersections) of these} \end{matrix} \right\}$$

which signifies that we have either *u, n* or *d* on the first-time step. 1.0 pt

$\Im_2 = \Im =$ The collection of all subsets of Ω. 0.25 pts

(d) For S_1, what matters are the outcomes at time 1, which are *u*, *n* and *d*; each of these outcomes need to be combined with possible outcomes at time 2. However, note that the calculation should only focus up to time 1 since outcomes at time 2 are still not revealed at time 1. So, we have

$S_1(uu) = S_1(un) = S_1(ud) = 50(12/11) = 54.55$
$S_1(nu) = S_1(nn) = S_1(nd) = 50(1) = 50.00$ 1 pt
$S_1(du) = S_1(dn) = S_1(dd) = 50(10/11) = 45.45$

Therefore, $S_1(\omega) = \begin{cases} 54.55 & if \quad \varpi = uu, un, ud \\ 50.00 & if \quad \varpi = nu, nn, nd \\ 45.45 & if \quad \varpi = du, dn, dd \end{cases}$ 1 pt

(e) We consider the interval $[2\pi^e - 1,\ 2e^\pi + 1]$ = [43.9183, 47.2814]. The pre-image under S_1 of the interval [43.9183., 47.2814] is defined to be $\{\omega \in \Omega : S_1(\omega) \in [43.9183, 47.2814]\}$ = $\{\omega \in \Omega : 43.9183 \le S_1 \le 47.2814\}$ = {*du, dn, dd*}.
0.5 pts
0.5 pts

♣Required Assignment Problem #4 [4 points]

Assume a single-period model with $r = 0.05$ and S(0) =100. After one period, the stock price can either go up to 101 or down to 99.

(a) Find a replicating portfolio for a put option with the pay off function *g(s)* = $(100\text{-s})^+$ = max(100-s, 0). **Use a replicating portfolio consisting of α amount held in a savings account and Δ shares of holdings in stock.** [3 pts] [**Note:** *You will get a zero for this question if you construct a different replicating portfolio.*]

(b) Using your replicating portfolio in (a), what is the price of the put option in question? [1 pt]

Answer:

Let α and Δ be the respective holdings in the money market account and stock investments.

(a) From the principle of replication, we have the system of equations below

$$1.05\alpha + 101\Delta = 0$$
$$1.05\alpha + 99\Delta = 1$$

2 pts

0.25 pts

Solving the above system yields $\alpha = 48.0952$ and $\Delta = -0.5$.

This means a short position in 0.5 shares and $48.0952 must be held in the money market account.

0.25 pts for interpretation

0.75pts

(b) The cost of setting up the portfolio is at time 0 is 100(-0.5) + 48.0952= -$1.9048. This is the cost for shorting the put (from the perspective of the company selling). The put price is therefore, $1.9048. 0.25 pts