Maths 190 Assignment No. 3 (Practice Exercises)

Do as indicated. ENJOY!

1. A company's cash position (in million of dollars) follows a generalised Wiener process with a drift of 0.5 per quarter and variance rate of 4.0 per quarter. How high does the company's initial position have to be for the company to have a less than 5% chance of negative cash position at the end of one year?

2. Variables X_1 and X_2 follow generalised Wiener proceses with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. What process does $X_1 + X_2$ follow if:
   a. The changes in X_1 and X_2 in any short time interval are uncorrelated?
   b. There is a correlation ρ between the changes in X_1 and X_2 in any short time interval?

3. Consider a variable S that follows the process $dS_t = \mu dt + \sigma dW_t$. For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year 6?

4. Suppose that a stock price S_t follows a geometric Brownian motion. Show that if $n \in \mathbb{R}$, the process S_t^n also follows a geometric Brownian motion.

5. Suppose that x_t is the yield to maturity with continuous compounding on a zero-coupon bond that pays \$1 at time T. Assume that x_t is stochastic and follows the dynamics

$$dx_t = a(x_0 - x_t)dt + s x_t dW_t$$

   where a, x_0 and s are positive constants and W_t is a Brownian motion. Determine the process followed by the bond price.

6. The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

7. Explain the principle of risk-neutral valuation.

8. Using the notation in the lecture, prove that if S_t has a lognormal dynamics then a 95% confidence interval for S_T is

$$\left(S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)-1.96\sigma\sqrt{T-t}},\ S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)+1.96\sigma\sqrt{T-t}}\right).$$

9. Consider a derivative that pays off S_T^n at time T, where S_T is the stock price at that time. When the stock price follows a geometric Brownian motion, it can be shown that its price at time t ($t \leq T$) has the form $h(t,T)S^n$ where S is the stock price at time t and h is a function only of t and T.
   a. By substituting into the Black-Scholes partial differential equation, derive an ordinary differential equation satisfied by $h(t,T)$.
   b. What is the boundary condition for the differential equation for $h(t,T)$?
   c. Show that $h(t,T) = \exp[0.5\sigma^2 n(n-1) + r(n-1)(T-t)]$ where r is the risk-free interest rate and σ is the stock price volatility.

10. "Once we know how to value options on a stock paying a dividend yield, we know how to value options on stock indices, currencies and futures." Explain this statement.

11. Show that if S_t is a geometric Brownian motion, i.e., $dS_t = \mu S_t dt + \sigma S_t dW_t$ then $S_t = S_0 e^{(\mu-\sigma^2/2)dt+\sigma W_t}$.

12. In the previous lecture, we show that if μ and σ are the respective drift and volatility of a stock price process following a geometric Brownian motion then the forward price evolves according to the stochastic dynamics $dF_t = (\mu - r)F_t dt + \sigma F_t dW_t$, where r is the risk-free rate. Prove or disprove that the discounted forward price process is a martingale? **Hint:** *Note that a driftless process is a martingale.*

13. Show that the Black-Scholes formulae for call and put options satisfy the put-call parity.

14. Show that the probability that a European call option will be exercised in a risk-neutral world, with the notation introduced in the lecture, is $\Phi(d_2)$. What is the expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K?

15. The price of a European call that expires six months and has a strike price of \$30 is \$2. The underlying stock price is \$29 and a dividend yield of \$0.50 and is expected in two months and again five months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put that expires in six months and has a strike price of \$30?

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