Maths 190
Solutions to Assignment No.3
Problems NOT required for submission

N.B.: For the Final Exam, make sure you know how to read the Table of Standard Normal Probabilities (which will be provided) to evaluate Brownian-motion and normal-probability type questions.

1. Suppose the company's initial cash position is x. The probability distribution of the cash position at the end of one year is $N(x+4(0.5), 4(4)) = N(x+2.0, 16)$. Thus, the probability of a negative cash position at the end of one year is $\Phi\left(-\frac{x+2.0}{4}\right)$ where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable. We have to find x such that $\Phi\left(-\frac{x+2.0}{4}\right) = 0.05$. From the normal distribution table, this happens when $-\frac{x+2.0}{4} = -1.6449$, i.e., $x = 4.5796$. The initial cash position must therefore be \$4.58 million.

2. a) Suppose that X_1 and X_2 equal a_1 and a_2 initially. After a time period of length T, X_1 has the probability distribution $N(a_1 + \mu_1 T, \sigma_1^2 T)$ and X_2 has a probability distribution $N(a_2 + \mu_2 T, \sigma_2^2 T)$. From the property of sums of independent normally distributed variables, $X_1 + X_2$ has the probability distribution $N(a_1 + \mu_1 T + a_2 + \mu_2 T, \sigma_1^2 T + \sigma_2^2 T)$, i.e., $N\left(a_1 + a_2 + (\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2)T\right)$. This shows that $X_1 + X_2$ follows a generalised Wiener process with drift rate $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

   b) In this case, since $\mu_1,\ \mu_2,\ \sigma_1,\ \sigma_2$ and ρ are all constant, this implies that the distribution of the sum $X_1 + X_2$ in a longer period of time T is
   $$N\left(a_1 + a_2 + (\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)T\right).$$
   The variable $X_1 + X_2$, therefore, follows a generalised Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$.

3. The change in S during the first three years has the probability distribution $N(2(3), 9(3)) = N(6, 27)$. The change during the next three years has the probability distribution $N(3(3), 16(3)) = N(9, 48)$.

The change during the six years is the sum of a variable with probability distribution $N(6, 27)$ and a variable with probability distribution $N(9, 48)$. The probability distribution of the change is therefore $N(15, 75)$. Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is $N(20, 75)$.

4. If $G(S,t) = S^n$ then $\dfrac{\partial G}{\partial t} = 0$, $\dfrac{\partial G}{\partial S} = nS^{n-1}$, and $\dfrac{\partial^2 G}{\partial S^2} = n(n-1)S^{n-2}$. Using Itô's lemma,

$$dG_t = \left(\mu n G + \frac{1}{2}n(n-1)\sigma^2 G\right) dt + \sigma n G dW_t.$$

This shows that $G = S^n$ follows a geometric Brownian motion where the expected return is $\mu n + \dfrac{1}{2}n(n-1)\sigma^2$ and the volatility is $n\sigma$. The stock price S has an expected return of μ and the expected value of S_T is $S_0 e^{\mu T}$. The expected value of S_T^n is $S_0^n e^{\left(\mu n + \frac{1}{2}n(n-1)\sigma^2\right)T}$.

5. The process followed by B, the bond price, is from Itô's formula given by

$$dB_t = \left(\frac{\partial B}{\partial x} a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}\right) dt + \frac{\partial B}{\partial x} sx dW_t.$$

Since $B = e^{-x(T-t)}$ the required partial derivatives are:

$$\begin{aligned}
\frac{\partial B}{\partial t} &= xe^{-x(T-t)} = xB \\
\frac{\partial B}{\partial x} &= -(T-t)e^{-x(T-t)} = -(T-t)B \\
\frac{\partial^2 B}{\partial x^2} &= (T-t)^2 e^{-x(T-t)} = (T-t)^2 B.
\end{aligned}$$

Hence,

$$dB_t = \left(-a(x_0 - x)(T-t) + x + \frac{1}{2}s^2x^2(T-t)^2\right) Bdt - sx(T-t)BdW_t.$$

6. The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and assuming 252 trading days in one year, $\Delta t = \dfrac{1}{252} = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

7. The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in the risk-neutral world as they do in the real world. We may therefore assume that the world is risk-neutral for the purpose of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to the risk-free rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

8. From the lecture it was shown that $\ln S_T \sim N\left(\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right)$. The respective left and right endpoints of a 95% confidence interval for $\ln S_T$ are therefore

$$\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t) - 1.96\sigma\sqrt{T-t}$$

and

$$\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)T + 1.96\sigma\sqrt{T-t}.$$

The respective left and right endpoints of a 95% confidence interval for S_T are therefore

$$e^{\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t) - 1.96\sigma\sqrt{T-t}}$$

and

$$e^{\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t) + 1.96\sigma\sqrt{T-t}}.$$

That is,

$$S_t e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) - 1.96\sigma\sqrt{T-t}}$$

and

$$S_t e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + 1.96\sigma\sqrt{T-t}}.$$

9. a) If $G(S,t) = h(t,T)S^n$ then $\frac{\partial G}{\partial t} = h_t S^n$, $\frac{\partial G}{\partial S} = hnS^{n-1}$ and $\frac{\partial^2 G}{\partial S^2} = hn(n-1)S^{n-2}$ where $h_t = \frac{\partial h}{\partial t}$. Substituting into the Black-Scholes differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh.$$

b) The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T, T) = 1$.

c) The equation $h(t, T) = e^{\left[0.5\sigma^2 n(n-1) + r(n-1)\right](T-t)}$ satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a) directly. The differential equation (DE) can be re-written as

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1).$$

The solution to this DE is

$$\ln h = \left(-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)\right) t + k,$$

where k is a constant. Since $\ln h = 0$ when $t = T$ it follows that

$$k = \left(r(n-1) + \frac{1}{2}\sigma^2 n(n-1)\right) T$$

so that

$$\ln h = \left(r(n-1) + \frac{1}{2}\sigma^2 n(n-1)\right)(T - t)$$

or

$$h(t, T) = e^{\left[0.5\sigma^2 n(n-1) + r(n-1)\right](T-t)}.$$

10. A stock index is analogous to a stock paying a continuous dividend yield, the dividend yield being the dividend yield on the index. A currency is analogous to a stock paying a continuous dividend yield, the dividend yield being the foreign risk-free interest rate. A futures contract is analogous to a stock paying a continuous dividend yield, the dividend yield being the domestic risk-free interest rate.

11. Consider the function $f(x, t) := S_0 e^{\{\sigma x + (\mu - \frac{1}{2}\sigma^2)t\}}$. So, $S_t = f(W_t, t)$. Then
$\dfrac{\partial f}{\partial t} = \left(\mu - \dfrac{1}{2}\sigma^2\right) f$, $\dfrac{\partial f}{\partial x} = \sigma^2 f$ and $\dfrac{\partial^2 f}{\partial x^2} = \sigma^2 f$.

From Itô's formula,

$$\begin{aligned}
dS_t &= df(W_t, t) \\
&= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dW_t)^2 \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) f dt + \sigma f dW_t + \frac{1}{2}\sigma^2 f dt \\
&= \mu S_t dt + \sigma S_t dW_t.
\end{aligned}$$

12. Hint: Apply Itō's lemma to $e^{-rt}F_t$ and show that the dt part vanishes.

13. From the Black-Scholes equations

$$p_t + S_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1) + S_t.$$

Since $1 - \Phi(-d_1) = \Phi(d_1)$ this is

$$Ke^{-r(T-t)}\Phi(-d_2) + S_t\Phi(d_1).$$

Also,

$$c_t + Ke^{-r(T-t)} = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) + Ke^{-r(T-t)}.$$

Since $1 - \Phi(d_2) = \Phi(-d_2)$, this is also

$$Ke^{-r(T-t)}\Phi(-d_2) + S_t\Phi(d_1).$$

14. The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk-neutral world,

$$\ln S_T \sim N(\ln S_t + (r - \sigma^2/2)(T-t), \sigma^2(T-t)).$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$1 - \Phi\left(\frac{\ln K - \ln S_t - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) = \Phi\left(\frac{\ln(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) = \Phi(d_2).$$

15. If D represents the present value of the dividends, then we have the put-call parity $c_t + Ke^{-r(T-t)} + D = p_t + S_t$ or $p_t = c_t + Ke^{-r(T-t)} + D - S_t$.

In this case, $p_t = 2 + 30e^{(-0.1)(6/12)} + 0.5e^{(-0.1)(2/12)} + 0.5e^{(-0.1)(5/12)} - 29 =$ \$2.51.

$\sim\sim\sim$ **E N D** $\sim\sim\sim$