Maths 190 (Math'l Methods in Finance) – Week of 06 – 10 February 2017

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts were covered/reviewed:

1. There are also Bermudan options that can only be exercised at specific dates known in advance.

2. An example of an Asian option, whose underlying's terminal value is based on average of prices in the past up to the stock's terminal price, was given in class. An Asian option is an example of an exotic options and it can either be European or American.

3. A stock index is an indicator that tracks the value of a hypothetical portfolio of stocks. Examples include S&P 500, Nikkei 225, NYSE Composite Index and the MMI.

4. Futures on stock indices are traded. In this case, a stock index can be regarded as the price of a security that pays dividends. The security is the portfolio of stocks underlying the index. If q is the dividend yield rate and S is the current value of the index then the futures price on a stock index is $F = Se^{(r-q)T}$.

5. Arbitrage arguments can be used to obtain exact futures prices in the case of investment commodities. However, **it turns out that they can only be used to give an upper bound to the futures price in the case of consumption commodities.**

   For instance, if $F > (S + U)e^{rT}$, an arbitrageur can buy the commodity (e.g., gold) and short a commodity futures contract to lock in a profit. If $F < (S+U)e^{rT}$, an arbitrageur who already owns a commodity can improve his/her return by selling the commodity and buying back the commodity via the terms of the futures contract.

6. Individuals and companies who keep commodities in inventory do so because of their consumption value – not because of their value as investments.

   The case $F > (S + U)e^{rT}$ cannot hold, or at least, it cannot hold for a significant amount of time. This is because an investor can (i) borrow an amount $S + U$ at the risk-free rate and use the loan to purchase one unit of the commodity and to pay the storage costs and (ii) short a futures contract on one unit of the commodity.

   On the other hand, when $F < (S+U)e^{rT}$, it is true that an investor can (i) sell the commodity, save the storage costs and invest the proceeds at the risk-free rate and (ii) buy the futures contract. However, investors may be reluctant to sell the commodities and buy futures contracts because *futures contracts cannot be consumed.* Thus, there is nothing to stop the inequality $F < (S + U)e^{rT}$ to hold.

   Thus, for futures on commodities for consumption, the most that we could obtain is

$$F \leq (S + U)e^{rT}.$$

7. Consider a **futures contract on commodities held solely for con-**

sumption. The most that we could obtain is

$$F \leq (S + U)e^{rT}. \qquad (1)$$

In the above, F, S, U, r and T are the forward price, spot price of the commodity, present value of the storage costs, risk-free rate and time to maturity, respectively.

8. When inequality (1) holds, users of the commodity must feel that there are benefits from ownership of the physical commodity that are not obtained by the holder of the futures contract. These may include: (i) the ability to profit from temporary local shortages and (ii) the ability to keep a production process running.

9. Benefits from holding commodities for consumption are sometimes referred to as *convenience yields* provided by the product. If the convenience yield is y (in percent), we showed in class that the futures price for consumption commodities is $F = Se^{(r+u-y)T}$. Note that convenience yield measures the extent to which the left-hand-side is less than the right-hand-side of inequality (1).

10. An example was provided in class demonstrating that convenience yield reflects the market's expectations concerning the future availability of the commodity. The greater the possibility that shortages will occur during the life of the futures contract, the higher the convenience yield. If users of the commodity have high inventories, there is very little chance of shortage in the near future and the convenience yield tends to be low. Low inventories, on the other hand, tend to lead to high convenience yield.

11. The relationship between futures price and spot prices can be summarised in terms of what is known as the *cost of carry.* Cost of carry

measures the storage cost plus the interest that is paid to finance the asset less the income earned on the asset. Using the previous notation, the cost of carry for non-dividend paying stock, stock index, currency and commodity with storage cost are r, $r-q$, $r-r_f$ and $r+u$, respectively.

12. Define the cost of carry as c. For an investment asset, the futures price is $F = Se^{cT}$. For a consumption asset, $F = Se^{(c-y)T}$.

13. Whereas a forward contract normally specifies that delivery is to take place on a particular day, a futures contract often allows the party with the short position to choose to deliver during a certain period of time. Consider the equation $F = Se^{(c-y)T}$. We see that if the futures price is an increasing function of the time to maturity T, benefits from holding the asset are less than the risk-free rate. It is then usually optimal for the party with the short position to deliver as early as possible. This is because the interest earned on the cash received outweighs the benefits of holding the asset. If the futures prices are decreasing as maturity increases, the reverse is true.

14. An important research question to address is whether the futures price is an unbiased estimate of the asset price at time T, i.e., is $F = E[S_T]$? Many empirical studies found that $F < E[S_T]$. If the capital asset pricing model (usually introduced in a 2nd-year corporate finance course) is true, the relationship between F and $E[S_T]$ *mainly* depends on whether the spot price is positively or negatively correlated with the level of the stock market.

15. An American or European call option gives the holder the right to buy one share of stock for a certain price in the future. No matter what

happens, the option can never be worth more than the stock. Therefore, $c \leq S$ and $C \leq S$, where c and C are the prices of European and American call options, respectively.

If the above relations are not true, an arbitrageur can easily make a riskless profit by buying the stock and selling the call option.

16. An American or European put option gives the holder the right to sell one share of a stock for X. No matter how low the stock price becomes, the option can never be worth more than X. Hence, $p \leq X$ and $P \leq X$, where where p and P are the prices of European and American put options, respectively. Again, if these relations are not true an investor can execute strategies that can lock in riskless profit.

    It was also demonstrated and argued that a tighter upper bound for p is Xe^{-rT}, i.e,. $p \leq Xe^{-rT}$.

17. We establish a lower bound for the price of a European call option on a non-dividend-paying stock, which is $S - Xe^{-rT}$. More formally, $c > \max(S - Xe^{-rT}, 0)$ since the call price is restricted to be positive. This result can be derived by considering two portfolios:
    Portfolio A: One European call option plus an amount of cash equal to Xe^{-rT}.

    Portfolio B: One share.

    It can be verified that Portfolio A is worth $\max(S_T, X)$ at time T. On the other hand, portfolio B is worth S_T at time T. Hence, portfolio A is always worth as much as, and is sometimes worth more than portfolio B at time T. Consequently, the present value of portfolio A is always worth as much as, and is sometimes worth more than the present value of portfolio B at time 0. That is,

    $$c + Xe^{-rT} > S \quad \text{or} \quad c > S - Xe^{-rT}.$$

    The worst that can happen to a call option is that it expires worthless, its value must be positive, i.e., $c > 0$, or $c > \max(S - Xe^{-rT}, 0)$.

18. For a European put option on a non-dividend-paying stock, a lower bound for the price is $Xe^{-rT} - S$. More precisely, $p > \max(Xe^{-rT} - S, 0)$. This could be established by considering two portfolios:

    Portfolio C: One European put option plus one share.

    Portfolio D: An amount of cash equal to Xe^{-rT}.

    One needs to show that portfolio C is always worth as much, and is sometimes worth more than portfolio D at time T.

19. We identified the factors that affect stock option prices. These include the following: (i) current stock price, S, (ii) strike price, X, (iii) time to expiration, T, (iv) volatility of the stock price, σ, (v) risk-free rate, r and (vi) dividends expected during the life of the option, with present value, D.

20. We showed that *it is never optimal to exercise an American call option on a non-dividend-paying stock early.* This can be established by considering two portfolios:
    Portfolio E: One American call option plus an amount of cash equal to Xe^{-rT}.

    Portfolio F: One share.

    Then, examine two cases: when the call option is exercised at time t, $0 < t < T$ and when the call option is held to expiration. In the first case, portfolio E, is always worth less than portfolio F. In the second case, it can be shown that portfolio E is always worth as much as, and is, sometimes worth more than portfolio F.

    Therefore, an American call option on a non-dividend-paying stock should never be exercised prior to the expiration date. This implies that an American call option on a non-dividend-paying stock is, therefore, worth the same as the corresponding European option on the same stock. Since $C = c$ then $C > S - Xe^{-rT}$.

21. There are two reasons why a call option on a non-dividend-paying stock should not be exercised early:
(i) It provides insurance. Once the option has been exercised and the exercise price has been exchanged for the stock price, this insurance vanishes.

    (ii) Time value of money. The later the strike price is paid out the better.

22. It can be optimal to exercise an American put option on a non-dividend-paying stock early. We demonstrated in class that at any given time during the life of a put option, it should always be exercised early if it is sufficiently deeply in-the-money.

23. For an American put with price P, the condition $P \geq X - S$ must always hold since immediate exercise is always possible.

24. Recall that the *intrinsic value* is the maximum of zero and the value of the option will have if it were exercised immediately. Since there are some circumstances when it is desirable to exercise an American put option early, it follows that an American put option is always worth more than the corresponding European put option. Also, since an American put is sometimes worth its intrinsic value, it follows that a European put option must sometimes be worth less than its intrinsic value.

25. The following relationships between American call and put prices were established:

$$P \geq C + Xe^{-rT} - S \quad \text{or} \quad C - P \leq S - Xe^{-rT},$$
$$C - P \geq S - X$$

and consequently

$$S - X \leq C - P \leq S - Xe^{-rT}.$$

26. Let D be the present value of dividends during the life of an option. Then we have the following results:

(a) We modify $c \geq S - Xe^{-rT}$ into $c \geq S - D - Xe^{-rT}$.

(b) We modify $p \geq Xe^{-rT} - S$ into $p \geq D + Xe^{-rT} - S$.

(c) The put-call parity is modified to $c + D + Xe^{-rT} = p + S$.

(d) The inequality $S - X \leq C - P \leq S - Xe^{-rT}$ is modified to $S - D - X \leq C - P \leq S - Xe^{-rT}$.

(e) If q is the dividend yield (expressed as % of the spot price) then we have the put-call parity

$$p + Se^{-qT} = c + Xe^{-rT}.$$

Also,

$$Se^{-qT} - X \leq C - P \leq S - Xe^{-rT}.$$

27. When dividends are expected, it is sometimes optimal to exercise an American call immediately prior to an ex-dividend date. This is because the dividend will cause the stock price to jump down making the option less attractive.

28. Pricing European options using binomial trees.
Let S =current stock price, u =“appreciation” factor when the stock price moves up and d =“depreciation” factor when the stock price moves down and so
$u-1$ =proportional increase when there is an up movement. The stock price goes up to the new level Su $(u > 1)$.
$1-d$ =proportional decrease when there is a down movement. The stock price goes down to the new level Sd $(d < 1)$.
r =risk-free rate.

By considering a riskless portfolio (consisting of a long position in Δ shares and a short position in one option), it was shown that $\Delta = \dfrac{f_u - f_d}{Su - Sd}$ where f_u =pay-off from the option when the stock price is Su and f_d =pay-off from the option when the stock price is Sd. Finally, we equate the cost of setting up the portfolio with its present value. If f denotes the price of an option then

$$f = e^{-rT}\left[qf_u + (1-q)f_d\right]$$

where $q = \dfrac{e^{rT} - d}{u - d}$.

The above argument can be extended to a two-step binomial pricing model, and in general to an $n-$step binomial pricing. As we increase n, the option pricing formula will converge to the Black-Scholes option pricing representation. This will be discussed further in the succeeding lectures.