Maths 190 (Math'l Methods in Finance) – Week of 13 – 17 March 2017

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts were covered/reviewed:

1. We started looking at how BM can be used as a model for stock price evolution. We also looked at the dynamics of a stochastic process driven by a Brownian motion by considering the so-called stochastic differential equation.

2. In the computation of $\int_0^t (dW_s)^2$, we considered the term

$$\lim_{\|\pi_n\|\to 0} \sum_{i=0}^{n-1} \left[W_{t_{i+1}} - W_{t_i}\right]^2,$$

where $\|\pi_n\|$ is the norm of the partition of $[0,t]$. This is called the quadratic variation of W_t.

3. If we follow the Newton-Leibniz calculus, $dW_t^2 = 2W_t dW_t$. We showed however that in stochastic calculus, this not the case as we would obtain $dW_t^2 = 2W_t dW_t + dt$. In integral form,

$$W_t^2 = 2\int_0^t W_s dW_s + t.$$

The second term t is the correction term coming from the quadratic variation. It has to be there so that both sides of the above equality (with integral on the right-hand side) have equal expectations.

Note that we proved $E\left[\int_0^t W_s dW_s\right] = 0$ by approximating the integral and using the fact that Brownian motion has stationary and independent increments.

4. Itō's lemma was derived on the basis of Taylor series expansion of a function. All higher-order terms after the second-order turn out to go to zero.

   In particular, this lemma states the following: Suppose X_t is a stochastic process whose dynamics is given by the SDE $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$. If $f \in C^{2,1}$, then $G_t := f(X_t, t)$ is also a stochastic process with dynamics given by

$$dG_t = \left(\frac{\partial f}{\partial t} + \mu(X_t, t)\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma(X_t, t)^2\right) dt + \frac{\partial f}{\partial x}\sigma(X_t, t)dW_t.$$

5. In developing Ito's lemma, with W_t being a BM, it was mentioned and heuristically explained that we have the "multiplication rule": $(dW_t)^2 = dt$, $(dt)^2 = 0$ and $(dW_t)(dt) = 0$.

6. Several examples were given, within the context of financial applications, to illustrate the workings of Itoō's lemma.