# Maths 190 (Math'l Methods in Finance) – Week of 27 February – 03 March 2017

## SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts were covered/reviewed:

1. Pricing European options using binomial trees under a one-period framework:
   Let $S$ =current stock price, $u$ =“appreciation” factor when the stock price moves up and $d$ =“depreciation” factor when the stock price moves down and so
   $u-1$ =proportional increase when there is an up movement. The stock price goes up to the new level $Su$ ($u > 1$).
   $1-d$ =proportional decrease when there is a down movement. The stock price goes down to the new level $Sd$ ($d < 1$).
   $r$ =risk-free rate.

   By considering a riskless portfolio (consisting of a long position in $\Delta$ shares and a short position in one option), it was shown that $\Delta = \dfrac{f_u - f_d}{Su - Sd}$ where $f_u$ =pay-off from the option when the stock price is $Su$ and $f_d$ =pay-off from the option when the stock price is $Sd$. Finally, we equate the cost of setting up the portfolio with its present value. If $f$ denotes the price of an option then

   $$f = e^{-rT}\left[qf_u + (1-q)f_d\right], \tag{1}$$

   where $q = \dfrac{e^{rT} - d}{u - d}$.
   The above argument can be extended to a two-step binomial pricing model, and in general to an $n-$step binomial pricing. As we increase $n$, the option pricing formula will converge to the Black-Scholes option pricing representation.

2. Pricing European options using binomial trees under a two-period framework:
For a two-step binomial pricing model, let
$f_u$=pay-off from the option when the stock price is $Su$
$f_{uu}$= pay-off from the option when the stock price is $Su^2$
$f_{ud}$=pay-off from the option when the stock price is $Sud = Sdu$
$f_d$=pay-off from the option when the stock price is $Sd$
$f_{dd}$=pay-off from the option when the stock price is $Sd^2$

Over one period, the option pricing formula is $f = e^{-rT}\left[qf_u + (1-q)f_d\right]$.
For a two-period model in which the length of one period is $\Delta T$ the price of a European option is $f = e^{-2r\Delta T}\left[q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}\right]$.

3. If we generalise the use of binomial trees by adding more steps to the tree, we find that the risk-neutral valuation principle continues to hold. That is, the option price is always equal to its expected pay-off in a risk-neutral world discounted at the risk-free rate; this is, assuming of course, that the interest rate is constant throughout the life of the option contract.

4. Procedures in pricing of American options using binomial trees:
($i$) Work backwards through the tree from the end to the beginning, testing at each node to see whether early exercise is optimal.
($ii$) The value of the option at the final nodes is the same as for the European option.
($iii$) At earlier nodes the value of the option is the maximum of
(a) $f = e^{-rT}\left[qf_u + (1-q)f_d\right]$
and
(b) the pay-off from early exercise.

The above procedure was illustrated using a numerical example in the lecture. It is apparent that if early exercise is optimal, the price of the American call/put is greater than the price of the corresponding European call/put.

5. In practice, the life of an option is typically divided into at least 30 time steps when implementing the binomial pricing method. This means that $2^{30}$ (approx 1 billion) possible stock price paths are considered.

6. An important parameter in the pricing and hedging of options is delta ($\Delta$). This parameter represents the number of units of the stock that one should hold for each option shorted in order to create a riskless hedge. More formally, it is the ratio of the change in the price of a stock option to the change in the price of the underlying stock.

7. The construction of a riskless hedge is referred to as *delta hedging*. The delta of a call is positive whilst the delta of a put is negative.

8. It was also demonstrated in the lecture that in order to maintain a riskless hedge using an option and the underlying stock, we need to adjust our holdings in the stock periodically.

9. The complete set of equations for the implementation of the binomial option pricing model is:
   $u = e^{\sigma\sqrt{\delta t}}$; $d = e^{-\sigma\sqrt{\delta t}}$ and $q = \dfrac{e^{r\delta t} - d}{u - d}$,
   where the notations employed above are defined in (1) and $\delta t$ is one-period time step.

10. When confronted with problems or questions involving options whose underlying variable is the price of futures contract or foreign exchange

rate, all we need to do is modify the risk-neutral probability $q$, i.e.,

$$q = \frac{e^{(r-g)T} - d}{u - d},$$

where $g$ is the income (expressed in % terms).

In particular, replace $g$ by $r$ for options on futures contract and replace $g$ by $r_f$ (foreign risk-free rate) when the underlying variable is a foreign exchange rate.

Note that the change from $r$ to $r - g$ hinges on the fact that an income (e.g, dividend) to the holder of the asset decreases the stock price.

## BASIC ELEMENTS OF STOCHASTIC PROCESSES RELEVANT TO DERIVATIVE PRICING

11. A stochastic process $S$ is a **martingale** with respect to a measure $Q$ and a filtration $\{\mathcal{F}_k\}$ if (i) $S_k$ is $\mathcal{F}_k-$adapted, (ii) $E^Q[S_k] < \infty$ and (iii) $E^Q[S_k|\mathcal{F}_j] = S_j, \quad \forall\ j \leq k$. It simply means that the future expected value at time $k$ of the process $S$ under $Q$ conditional on its history until time $j$ is merely the process value at time $j$.

    An example was given in the lecture.

12. We may occasionally need to use the fact that for $i \leq j$ and claim $H$,

    $$E^P\left[E^P[H|\mathcal{F}_j]|\mathcal{F}_i\right] = E^P[H|\mathcal{F}_i].$$

    In other words, conditioning on the history up to time $j$ and then conditioning on the history up to earlier time $i$ is the same as just conditioning originally up to time $i$. This result is called the **tower law**.

13. We showed in class that for any claim $H$, ***the conditional expectation process*** $E^P[H|\mathcal{F}_i]$ ***is always a*** $P-$***martingale using the***

***tower law.***

14. The concepts of probability space, stochastic basis or filtered probability space, $\sigma-$field, and probability measure were defined **formally** in class.

15. Hints and examples were given to help solve Assignment No. 2.

**RANDOM WALK**

16. We considered the random walk process under some measure, say, $P$. In particular, for $n \in \mathbb{Z}^+$, the random walk is a binomial process $W_n(t)$ satisfying the following conditions: (i) $W_n(0) = 0$, (ii) layer spacing of $\frac{1}{n}$, (iii) up and down jumps equal and of equal size $\frac{1}{\sqrt{n}}$ and (iv) under measure $P$, the up and down probabilities everywhere are equal to $\frac{1}{2}$.

    If $X_1,\ X_2,\ \ldots$ is a sequence of independent binomial RVs taking values +1 and -1 with equal prob, then value of $W_n$ at the $ith$ step has the representation

$$W_n\left(\frac{i}{n}\right) = W_n\left(\frac{i-1}{n}\right) + \frac{X_i}{\sqrt{n}}, \quad \text{for} \;\; i \geq 1.$$

17. We showed in class that $W_n(1)$ is normal with mean 0 and variance 1. In general, by using the central limit theorem, we showed that $W_n(t) \sim N(0, t)$ as $n \to \infty$.

18. Random walk has the property that its future movements away from a particular position are independent of where that position is and indeed, independent of its entire history of movements up to that time.

Moreover, the increment/displacement $W_n(s+t) - W_n(s)$ has mean 0 and variance $t$. Consequently, $W_n$ converges towards a Brownian motion. These observations on $W_n(s+t) - W_n(s)$ form the basis of the definition for the Brownian motion below.

19. The process $W = \{W_t : t \geq 0\}$ is a $P-$Brownian motion (BM) if (i) $W_t$ is continuous and $W_0 = 0$, (ii) $W_t \sim N(0,t)$ under $P$ and (iii) $W_{t+s} - W_s \sim N(0,t)$ under $P$ and independent of $\mathcal{F}_s$, the history of what the process attains up to time $s$.

20. BM is also called Wiener process. It is a one-dimensional Gaussian process.

21. Although a BM $W$ is continuous everywhere, it is nowhere differentiable. BM will eventually hit any value no matter how large or negative. It may be a million units above the horizontal axis but it will (with probability one) be back again to zero at some later time. Once a BM hits a value, it hits again infinitely often, and then again from time to time in the future.

22. We started looking at how BM can be used as a model for stock price evolution. We also looked at the dynamics of a stochastic process driven by a Brownian motion by considering the so-called stochastic differential equation.