# Maths 190 (Math'l Methods in Finance) – Week of 30 January – 03 February 2017

## SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts were covered/reviewed:

1. There are two sides to every option contract. One side is the investor who has taken the long position (i.e., has bought the option). On the other side is the investor who has taken a short position (i.e., has sold or *written* the option). The writer of an option receives cash up front but has potential liabilities later. His or her profit/loss is the reverse of that for the purchaser of options. There are four types of option positions: (i) a long position in a call option, (ii) a long position in a put option, (iii) a short position in a call option and (iv) a short position in a put option.

2. It is often useful to characterise European option positions in terms of the terminal value or pay-off to the investor at maturity. The initial cost (called the premium) is not included in the calculation of the pay-off. If it is included (without taking into account interest rate, i.e., no discounting/compounding is involved for simplicity of analysis) then one considers the profit.

   If $X$ is the strike price and $S_T$ is the final price of the underlying asset, the pay-off from a long position in a European call is $\max(S_T - X, 0)$. This reflects the fact that the option will be exercised if $S_T > X$ and will not be exercised if $S_T \leq X$.

   The pay-off to the holder of a short position in the European call option is $-\max(S_T - X, 0) = \min(X - S_T, 0)$.

   The pay-off to the holder of a long position in a European put is

$\max(X - S_T, 0)$ and the pay-off from a short position in a European put option is $-\max(X - S_T, 0) = \min(S_T - X, 0)$.

Sketches of pay-off and profit diagrams for these four option positions were given in the lecture.

3. Let $F$ :=forward price, $S$ :=spot price, $r$ :=risk-free rate and $T$ :=time to maturity of the forward contract. We saw in class that the "fair" price of a forward contract is given by $F = S(1+r)^T$. This assumes that $r$ is compounded annually. Any price greater or less than $F$ leads to an arbitrage opportunity.

   If $F$ is too high (i.e., $F > S(1+r)^T$) then an investor can do the following: (i) Loan an amount $S$ at the rate $r$; use the loan to buy one unit of the underlying asset and (ii) Enter into a short forward contract to sell the asset for $F$ at time $T$.

   If $F$ is too low (i.e., $F < S(1+r)^T$) then an investor can do the following: (i) Sell the asset for $S$. Invest the proceeds of the sale at the rate $r$ and (ii) Enter into a long forward contract to buy (back) the asset at time $T$.

   We demonstrated the above strategies in determining the price of a forward contract with gold as the underlying asset.

4. Futures markets were originally set up to meet the needs of hedgers. Farmers wanted to lock in an assured price for their produce. Merchants wanted to lock in a price they would pay for this produce. Futures contracts enabled both sides to achieve their objectives. Commodity futures are still widely used by producers and users of commodities for hedging. Financial futures can also be used for hedging purposes. We illustrated this by considering a company based in the United States knowing that it will have to pay £ 1 million in the future for goods it has purchased from a British supplier. With hedging, the cost of, or

price received for, the underlying asset is assured. However, there is no guarantee that the outcome with hedging will be better than the outcome without hedging.

5. Options can also be employed for hedging. We considered an investor who owns a number of shares of one stock and is concerned that the share price may decline sharply in the future. He or she wishes to protect himself or herself from this decline.The investor could then buy put options. The strategy will entail a cost but it will guarantee that each share can be sold at least at the strike price specified in the option.

6. There is a fundamental difference between the use of futures and options for hedging. Futures contracts are designed to neutralise the risk by fixing the price that the hedger will pay or receive for the underlying asset. Options contracts by contrast provides insurance. They provide a way in which investors can protect themselves against adverse price movements in the future whilst still allowing them to benefit from favourable price movements. Unlike futures, options will involve the payment of an up-front fee.

7. We considered an example (call option on Exxon) of how speculator could use options. The discussion and calculations in the example showed that options like futures provide a form of leverage. For a given investment, the use of options magnifies the financial consequences. Good outcomes become very good, whilst bad outcomes become very bad.

8. Further examples were provided to show how traders can take advantage of arbitrage opportunity by taking two or more offsetting positions in different markets. If, for instance, the futures price of an asset is getting out of line with the cash price, arbitrageurs will take offsetting

positions in the two markets to lock in a profit.

9. We analysed the relation between forward/futures prices and the price of the underlying asset. Forward contracts are easier to study because there is no daily settlement, i.e., only a single payment at maturity. Arguments were provided to derive results for prices of forward contracts on (i) securities providing no income, (ii) securities providing a known cash income and (iii) securities providing a known dividend yield.

10. Dividend–paying stocks and zero-coupon bonds are examples of securities that provide the holder with no income. Consider a forward contract on *a security with price $S$ that provides no income*. Let $T$ be the time to maturity, $r$ be the risk-free rate and $F$ be the forward price. Imagine an investor adopting the following strategy: (a) Buy one unit of the security (b) Short one forward contract. The forward contract has zero value at the time it is first entered into and therefore, the upfront cost of the strategy is just $S$. The forward contract requires the security to be exchanged for the forward price at time $T$.

    Thus, note that by following the strategy, the investor is simply exchanging a payment of $S$ today for a riskless cash inflow equal to the forward price at time $T$. Consequently, the forward price, $F$, must be the value to which $S$ would grow if invested at the risk-free rate for a time $T$. This implies that $F = Se^{rT}$. This argument was diagrammed in the lecture.

11. Consider a forward contract on *a security that provides income with a present value $I$*. Suppose an investor adopts the following strategy: (a) Buy the security (b) Enter into a short forward contract. The upfront cost of the strategy is the price of the security, $S$. The strategy provides the investor with income that has a present value (PV) of $I$ and a cash flow at $T$ equal to the forward price of the bond, $F$. Equating the initial outflow withe the PV of the cash inflows, we have $S = I + Fe^{-rT}$.

In other words, $F = (S - I)e^{rT}$.

12. Both currencies and stock indices can be regarded as *securities that provide known dividend yield*. A known dividend yield means that the income when expressed as a percentage of the security price is known. Consider an investor adopting the following strategy: (a) Buy $e^{-qT}$ of the security with income from the security being reinvested in the security (b) Short a forward contract.

    The holding of the security grows at rate $q$ so that $e^{-qT} \times e^{qT}$ or exactly one unit of the security is held at time $T$. Under the terms of the forward contract, the security is sold for $F$ at time $T$. The strategy, therefore, leads to an initial outflow of $Se^{-qT}$ and final inflow of $F$. The PV of the inflow must equal the outflow. Therefore, $Se^{-qT} = Fe^{-rT}$ or $F = Se^{(r-q)T}$.

13. We considered an example (call option on Exxon) of how speculator could use options. The discussion and calculations in the example showed that options like futures provide a form of leverage. For a given investment, the use of options magnifies the financial consequences. Good outcomes become very good, whilst bad outcomes become very bad.

14. The value of a forward contract at the time it is first entered into is zero. At a later time it may prove to have a positive or negative value. The value of a long forward contract, $f$, in terms of the originally negotiated price $K$, and the **current** forward price $F_0$ is given by

$$f = (F_0 - K)e^{-rT}.$$

    To see why the above equation is true, **compare a long forward contract with delivery price $F_0$ with an otherwise identical long forward contract that has a delivery price of $K$.** The difference between the two is only in the amount that will be paid for the underlying asset at time $T$. Under the first forward contract this amount is $F_0$; under the second contract it is $K$. The difference of $F_0 - K$ at time $T$ translates to an amount of $(F_0 - K)e^{-rT}$ today.

Recall again that the value of the contract that has a delivery price of $F_0$ is by definition zero. It follows that the value of the contract with a delivery price of $K$ is $(F_0 - K)e^{-rT}$.

The value of a forward contract on a security that provides no income is $f = S_0 - Ke^{-rT}$ since $F_0 = S_0e^{rT}$.

The value of a forward contract on a security that provides a known income with PV $I$ is $f = S_0 - I - Ke^{-rT}$ since $F_0 = (S_0 - I)e^{rT}$.

Lastly, the value of a forward contract on a security that provides a known dividend yield at rate $q$ is $f = S_0e^{-qT} - Ke^{-rT}$ since $F_0 = S_0e^{(r-q)T}$.

N.B.: In each case, the forward price, $F$, is the value of $K$ which makes $f$ equal zero when the contract was first entered into.

15. An example illustrating how to value a forward contract relying on the valuation formula, $f = (F_0 - K)e^{-rT}$ or $f = S_0 - Ke^{-rT}$, was given in the lecture.

16. Futures/forward contracts on currencies: Let $S$ be the current price in dollars of one unit of the foreign currency. A foreign currency has the property that the holder of the currency can earn interest at the risk-free interest rate prevailing in the foreign country. The holder, for example, can invest the currency in a foreign-denominated bond or a foreign-denominated savings account. Define $r_f$ as the value of the foreign risk-free rate with continuous compounding. Consider an investor adopting the following strategy: (a) Buy $e^{-r_fT}$ of the foreign currency

(b) Short a forward contract on one unit of the foreign currency. The holding in the foreign currency grows to one unit at time $T$ because of the interest earned. Under the terms of the forward contract, this is exchanged for $F$ at time $T$. The strategy, therefore, leads to an initial outflow of $Se^{-r_f T}$ and final inflow of $F$. The PV of the inflow must equal the outflow. Therefore, $Se^{-r_f T} = Fe^{-rT}$ or $F = S^{(r-r_f)T}$, which is the well-known interest-rate parity relationship from the field of international finance. A foreign currency is analogous to a security paying a known dividend yield. The "dividend yield", $q$, is the risk-free rate of interest in the foreign currency, $r_f$. Hence,

$$F = S^{(r-r_f)T}.$$

The above is the well-known interest-rate parity relationship from the field of international finance.

17. Futures contracts on the following currencies are trading in the IMM: Japanese yen, Canadian dollar, British pound and Australian dollar. In the case of Japanese yen, prices are expressed as the number of cents per unit of foreign currency. In the case of other currencies, prices are quoted as the number of US dollars per unit of foreign currency.

    Note that spot and forward rates are expressed as the number of units of the foreign currency per US dollar. So, a forward quote on the Canadian dollar of 1.200 would become a futures quote of 0.8333.

18. It is important to distinguish between commodities that are held solely for investment (e.g., gold and silver) and those that are held primarily for consumption.

19. For investment commodities, if storage costs are zero, they can be regarded as being analogous to securities paying no income. Note that storage costs can be viewed as negative income. So, if $S$ is the current

spot price of the commodity, $r$ is the risk-free rate, $T$ is the time to maturity, $U$ is the present value of all storage costs that will be incurred during the life of a futures contract then $F = (S + U)e^{rT}$. If $u$ is the storage costs per annum as a proportion of the spot price then $F = Se^{(r+u)T}$. If these relations do not hold, an investor can carry out strategies that would take advantage of arbitrage opportunities.

Arbitrage arguments can be used to obtain exact futures prices in the case of investment commodities. However, **it turns out that they can only be used to give an upper bound to the futures price in the case of consumption commodities.**

20. For investment commodities, if storage costs are zero, they can be regarded as being analogous to securities paying no income. Note that storage costs can be viewed as negative income. So, if $S$ is the current spot price of the commodity, $r$ is the risk-free rate, $T$ is the time to maturity, $U$ is the present value of all storage costs that will be incurred during the life of a futures contract then $F = (S + U)e^{rT}$. If $u$ is the storage costs per annum as a proportion of the spot price then $F = Se^{(r+u)T}$. If these relations do not hold, an investor can carry out strategies that would take advantage of arbitrage opportunities.