Financial Modelling 9561B
Winter 2014

Assignment No.1

GUIDELINES ON SUBMITTING ASSIGNMENTS

- Your assignment paper must include the Marking Scheme as a cover page. This marking scheme can be downloaded from the course website. **Failure to follow this instruction can result to a 2-point deduction on your assignment mark.**

- YOU MUST WRITE YOUR OWN WORK IN YOUR OWN WORDS, using full sentences and proper English grammar. It is your responsibility to familiarise yourself with the provisions of the University Regulation concerning academic integrity and honesty. **Any behaviour that can potentially lead to plagiarism, cheating and copying from/sharing with another student answers in an assignment or exams is a serious offence and carries with it severe penalty.** Do not take this warning lightly; academic penalties have dire consequences on your future studies and career.

- Do not submit your rough work! Do the problem set and then re-write it at least once - neatly, with adequate amount of clear explanation. The rewriting stage is the most important one for finding errors in one's work, and it will also deepen your understanding of the subject matter. Assignments are marked for both technical correctness and elegance of presentation.

- Bear in mind to include a sufficient amount of explanation about your work so that any marker does not have to guess what you mean. The grader of your work will determine if you understand what you are writing, not merely that you reach the particular correct answer.

- On questions where a computer output is required or deemed necessary, include the output in the text of your answer at the appropriate locations - do not put it all in a bunch at the end of your assignment. Unless, you are instructed to submit your work in a CD or disc, you are expected to hand in a PRINTED COPY.

Do as indicated. ENJOY!

Note: All questions involving W_t assumed that it (W_t) is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

1. When defining the Itō integral, the integrand is evaluated at the left endpoint of the interval. Suppose we consider the integral where the integrand is evaluated at the right endpoint instead. We shall call this the "backward" stochastic integral.
   (i) In particular, find the limit of the "backward" stochastic integral $\int_0^1 W_s dW_s$ by evaluating

   $$L^2 - \lim_{||\pi_n|| \to 0} \sum_{i=1}^{n} W\left(\frac{i}{n}\right)\left[W\left(\frac{i}{n}\right) - W\left(\frac{i-1}{n}\right)\right],$$

   where $||\pi_n||$ denotes the norm of a partition. [4 points]

   (ii) Offer an explanation why we use the left endpoint for the evaluation of the Itō integral in the context of viewing it as the gain/loss from trading. Equivalently, argue why the "backward" stochastic integral is not suited for financial applications unlike the Itō integral. Limit your answer to 15 words. [1 point]

2. Suppose X_t and $Y_t \in \mathcal{M}$, which is the class of integrands for which the Itō integral is well-defined. Show using linearity and martingale properties of an Itō integral along with the elementary identity $ab = \frac{(a+b)^2 - (a-b)^2}{4}$ (or otherwise) that

   $$E\left[\int_0^T X_t dW_t \int_0^T Y_t dW_t\right] = E\left[\int_0^T X_t Y_t dt\right].$$

   Make sure each step of your calculation is justified. [4 points]

3. Let X be a random variable on some $(\Omega, \mathcal{F}, P)$.
   (i) For $0 < r < s$, show that $E[|X|^r] \leq (E[|X|^s])^{r/s}$. [2 points] (Hint:

Recall Hölder's Inequality.)

(ii) Using 3(i) or otherwise (you may use the above result even if you are unable to prove it), prove $E\left|\int_0^t W_s^2 dW_s\right| \leq t^{3/2}, \quad t \geq 0$. [4 points]

4. Verify whether or not $X_t = \tan\left(\frac{\pi}{4} + W_t\right)$ is a solution of the SDE $dX_t = (1 + X_t^2)(X_t dt + dW_t)$. If it is a solution what is the appropriate initial value? [4 points]

5. Let $X_t, \quad t \geq 0$ be a solution of the SDE $\frac{dX_t}{X_t} = dt + dW_t$ with $X_0 = 1$. Is the random process

$$M_t := X_t^3 - 6\int_0^t X_s^3 ds, \quad t \geq 0$$

a martingale? Justify your answer. [4 points]

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