FM 9561B

Solutions to Assignment No. 1

Winter 2014

4 PTS

#1.(a) $\displaystyle\sum_{i=1}^{n} W\left(\frac{i}{n}\right)\left[W\left(\frac{i}{n}\right)-W\left(\frac{i-1}{n}\right)\right]$

1 $\displaystyle = \sum_{i=0}^{n-1} W\left(\frac{i+1}{n}\right)\left[W\left(\frac{i+1}{n}\right)-W\left(\frac{i}{n}\right)\right]$

$\displaystyle = \sum_{i=0}^{n-1} W_{\frac{i+1}{n}}\,\Delta W_{\frac{i}{n}}$, where $\Delta W_{\frac{i}{n}} = W\left(\frac{i+1}{n}\right)-W\left(\frac{i}{n}\right)$. (*)

1 $\displaystyle = \sum_{i=0}^{n-1}\left(\Delta W_{\frac{i}{n}} + W_{\frac{i}{n}}\right)\Delta W_{\frac{i}{n}}$ using (*).

$\displaystyle = \sum_{i=0}^{n-1}\left(\Delta W_{\frac{i}{n}}\right)^2 + \sum_{i=0}^{n-1} W_{\frac{i}{n}}\,\Delta W_{\frac{i}{n}}$

1 $\displaystyle \to n\cdot\frac{1}{n} + \int_0^t W_t\,dW_t = 1 + \int_0^1 W_t\,dW_t$.

as $\|\pi_n\| \to 0$.

In class, we showed $\displaystyle\int_0^t W_s\,dW_s = \frac{W_t^2}{2} - \frac{t}{2}$.

1 So, $\displaystyle\int_0^1 W_s\,dW_s = \frac{W_1^2}{2} - \frac{1}{2}$.

Finally, $\displaystyle L^2\text{-}\lim_{\|\pi_n\|\to 0} \sum_{i=0}^{n-1} W\left(\frac{i}{n}\right)\left[W\left(\frac{i}{n}\right)-W\left(\frac{i-1}{n}\right)\right]$

$\displaystyle = 1 + \frac{W_1^2}{2} - \frac{1}{2} = \frac{W_1^2}{2} + \frac{1}{2} = \frac{1}{2}(W_1^2+1)$.

(0.5)

(b) $W(\frac{i}{n})\left[W(\frac{i}{n}) - W(\frac{i-1}{n})\right]$ cannot be gain/loss from trading since $W(\frac{i}{n})$ is NOT ADAPTED at the beginning of the interval. (0.5)

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#2. $E\left[\int_0^T X_t\, dW_t \int_0^T Y_t\, dW_t\right]$

(2)

$$= E\left[\frac{\left(\int_0^T X_t\, dW_t + \int_0^T Y_t\, dW_t\right)^2 - \left(\int_0^T X_t\, dW_t - \int_0^T Y_t\, dW_t\right)^2}{4}\right],$$

using the identity $ab = \frac{(a+b)^2 - (a-b^2)}{4}$

with $a = \int_0^T X_t\, dW_t$ and $b = \int_0^T Y_t\, dW_t$

(1)

$$= \frac{E\left(\int_0^T (X_t + Y_t)\, dW_t\right)^2 - E\left(\int_0^T (X_t - Y_t)\, dW_t\right)^2}{4},$$

using the linearity property of the Itô integral and expectation operator

$$= \frac{E\left[\int_0^T (X_t + Y_t)^2\, dt\right] - E\left[\int_0^T (X_t - Y_t)^2\, dt\right]}{4}$$

by Itô's isometry

(1) by linearity property of Itô's integral

$$= E\left[\int_0^T \frac{(X_t + Y_t)^2 - (X_t - Y_t)^2}{4}\, dt\right] = E\left[\int_0^T X_t Y_t\, dt\right]$$

using the identity with $a = X_t$ and $b = Y_t$.

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3 (i) First, we recall Hölder's inequality:

1 Let X and Y be RVs on $(\Omega, \mathcal{F}, P)$, and let $p, q \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Then $E[|XY|] \leq \left(E[|X|^p]\right)^{1/p} \left(E|Y|^q\right)^{1/q}$.

Now, with $0 < r < s$, choose $p = s/r$ and

$$q = \frac{p}{p-1} = \frac{s/r}{s/r - r/r} = \frac{s/r}{\frac{s-r}{r}} = \frac{s}{s-r}$$

1 Applying Hölders' inequality to $|X|^r$ and $Y = 1_\Omega$, we get

$$E[|x|^r] \leq \left(E[(|x|^r)^{s/r}]\right)^{r/s} \cdot \left(E[|1_\Omega|^{s/s-r}]\right)^{s-r/s}$$

$$= \left(E[|x|^s]\right)^{r/s}.$$

(ii) With $r=1$ and $s=2$, we have from (i)

0.5 $$E[|X|] \leq \left(E[|x|^2]\right)^{1/2} = \sqrt{E[X^2]}$$

0.5 $$\therefore E\left[\left|\int_0^t W_s^2 \, dW_s\right|\right] \leq \left(E\left[\left(\int_0^t W_s^2 \, dW_s\right)^2\right]\right)^{1/2}$$

0.5 $$= \left(E\left[\int_0^t W_s^4 \, ds\right]\right)^{1/2} = \left(\int_0^t E[W_s^4] \, ds\right)^{1/2} \quad (*)$$

by Itô's isometry by Fubini's thm.

In equation (*), we need $E[W_t^4]$.

Let $f(t,x) = x^4$.

$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 4x^3, \quad \frac{\partial^2 f}{\partial x^2} = 12x^2.$ So, by

Ito's formula,

$$df(t, W_t) = 6W_t^2\, dt + 4 W_t^3\, dW_t, \quad W_0^4 = 0$$

(2)

In integral form,

$$W_t^4 = 0 + 6\int_0^t W_s^2\, ds + 4\int_0^t W_s^3\, dW_s.$$

$$\Rightarrow E[W_t^4] = 6\int_0^t E[W_s^2]\, ds + 0,$$ by the martingale property of the Ito's integral.

But $E[W_s^2] = s$ since $W_s \sim N(0, s)$

$$\therefore E[W_t^4] = 6\int_0^t s\, ds = 3t^2.$$

Returning to equation (*), we have

$$E\left[\left|\int_0^t W_s^2\, dW_s\right|\right] \le \left(\int_0^t E[W_s^4]\, ds\right)^{1/2}$$

$$= \left(\int_0^t 3s^2\, ds\right)^{1/2} = \left(t^3\right)^{1/2} = t^{3/2}.$$

0.5

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4. Consider $f(t,y) = \tan(\pi/4 + y)$

$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial y} = \sec^2(\pi/4 + y), \quad \frac{\partial^2 f}{\partial y^2} = 2\sec(\pi/4+y) \cdot \sec(\pi/4+y) \cdot \tan(\pi/4+y).$

Then,

$X_t = f(t, W_t) = \tan(\pi/4 + W_t).$

(1) Note: $1 + \tan^2(\pi/4 + y) = \sec^2(\pi/4 + y)$.

So, by Itô's lemma,

(1)
$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial y} dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dW_t)^2$$

$$= 0 + \sec^2(\pi/4 + W_t)\, dW_t + \cancel{\tfrac{1}{2}} \cdot \cancel{2} \sec^2(\pi/4 + W_t)\tan(\pi/4 + W_t)\, dt$$

(1)
$$= \left[1 + \tan^2(\pi/4 + W_t)\right] dW_t + \left[\left(1 + \tan^2(\pi/4 + W_t)\right)\tan(\pi/4 + W_t)\right] dt$$

(1)
$$= (1 + X_t^2)\, dW_t + \left[(1 + X_t^2) X_t\right] dt$$

$$= (1 + X_t^2)\left[dW_t + X_t\, dt\right].$$

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5. Consider $Y_t = X_t^3$.

Write $Y(t,x) = x^3$.

(1)
$$\frac{\partial Y}{\partial t} = 0$$
$$\frac{\partial Y}{\partial x} = 3x^2$$
$$\frac{\partial^2 Y}{\partial x^2} = 6x$$

NOTE: $dX_t = X_t\, dt + X_t\, dW_t$

$$dY_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{1}{2}\frac{\partial^2 f}{\partial x}(dx)^2$$

(1)
$$= 0 + 3X_t^2 (X_t\, dt + X_t\, dW_t) + \frac{1}{2} \cdot 6X_t (X_t^2\, dt)$$

$$= 3X_t^3\, dt + 3X_t^3\, dW_t + 3X_t^3\, dt$$

$$= 6X_t^3\, dt + 3X_t^3\, dW_t.$$

Thus, we obtain

$$dY_t - 6X_t^3\,dt = 3X_t^3\,dW_t$$

OR

$$Y_t - Y_0 - 6\int_0^t X_s^3\,ds = \int_0^t 3X_s^3\,dW_s$$

But $X_t^3 = Y_t$ and $X_0 = 1$.

Hence

$$Y_t - 1 - 6\int_0^t X_s^3\,ds = \int_0^t 3X_s^3\,dW_s$$

OR

$$X_t^3 - 6\int_0^t X_s^3\,ds = 1 + \int_0^t 3X_s^3\,dW_s \quad (\ast)$$

The RHS of $(\ast)$ is a stochastic integral and therefore, it is a martingale.

Clearly, the LHS, which is $X_t^3 - 6\int_0^t X_s^3\,ds$ is then a martingale.