SOLUTIONS TO ASSIGNMENT NO. 2 FM 9561B
Winter 2014

4 PTS

PROBLEM NO. 1 Consider the process X with SDE

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t,$$

$X_t = x$. Feynman-Kac's theorem states that

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if $\dfrac{\partial F}{\partial t}(t,x) + \mu(t,x)\dfrac{\partial F}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2(t,x)\dfrac{\partial^2 F}{\partial x^2}$

$-rF(t,x) = 0$ and $F(T,x) = \Phi(x)$

then $F(t,x) = E_{t,x}\left[e^{-r(T-t)}\Phi(X_T)\right]$.

From the PDE under consideration,

$\mu(t, X_t) = \mu X_t$ and $\sigma(t, X_t) = \sigma X_t$

1 and $r = 0$.

So, by the Feynman-Kac's result,

$$F(t,x) = E\left[e^{0(T-t)}\Phi(X_T)\,\middle|\, X_t = x\right].$$

1 Since we are given that $dX_t = \mu X_t\,dt + \sigma X_t\,dW_t$,

$$X_T = x\exp\left\{\left(\mu - \sigma^2/2\right)(T-t) + \sigma\left(W_T - W_t\right)\right\}.$$

$$X_T^2 = x^2\exp\left\{2\left(\mu - \sigma^2/2\right)(T-t) + 2\sigma\left(W_T - W_t\right)\right\}$$

$$\therefore\ F(t,x) = E\left[\ln X_T \,\middle|\, X_t = x\right]$$

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$$= E\left[\ln x + 2\left(\mu - \sigma^2/2\right)(T-t) + 2\sigma\left(W_T - W_t\right)\right]$$

$$= \ln x + 2\left(\mu - \sigma^2/2\right)(T-t) = \ln x + \left(2\mu - \sigma^2\right)(T-t)$$

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PROBLEM NO. 2 $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$

and $dZ_t = \Delta(t)\,dS_t + r_t\left(Z_t - \Delta(t)S_t\right)dt$ (1)

from previous lecture.

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Also $\beta(t) = \exp\left(\int_0^t r_u du\right)$ and $d\left(\frac{1}{\beta(t)}\right) = \frac{-r_t}{\beta(t)} dt$. (2)

$$d\left(\frac{S_t}{\beta(t)}\right) = \frac{1}{\beta(t)}\left[dS_t - r_t S_t dt\right] \quad (*)$$

also from the lecture.

Now, $d\left(\frac{Z_t}{\beta(t)}\right) = \frac{1}{\beta(t)} dZ_t + Z_t\, d\left(\frac{1}{\beta(t)}\right)$ as argued in class OR by Itô's lemma where $\left\langle Z_t, \frac{1}{\beta(t)}\right\rangle = 0$.

(1)

$$= \frac{1}{\beta(t)}\left[\Delta(t) dS_t + r_t (Z_t - \Delta(t) S_t) dt\right] + Z_t\left(\frac{-r_t}{\beta(t)}\right) dt$$

using the dynamics of Z_t and $\frac{1}{\beta(t)}$ in (1) and (2), respectively.

So, $d\left(\frac{Z_t}{\beta(t)}\right) = \frac{\Delta(t)}{\beta(t)} dS_t + \cancel{\frac{r_t Z_t}{\beta(t)}} dt$

(1)

$$- \frac{r_t}{\beta(t)} \Delta(t) S_t dt - \cancel{\frac{r_t Z_t}{\beta(t)}} dt$$

$$= \Delta(t) \cdot \frac{1}{\beta(t)}\left(dS_t - r_t S_t dt\right)$$

$$= \Delta(t)\, d\left(\frac{S_t}{\beta(t)}\right) \text{ using } (*).$$

Consider $d\left(\frac{S_t}{\beta(t)}\right) = \frac{1}{\beta(t)}\left[dS_t - r_t S_t dt\right]$

$$= \frac{1}{\beta(t)}\left[(\mu_t - r_t) S_t dt + \sigma_t S_t dW_t\right] \text{ because}$$

$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$.

Thus,

$$d\left(\frac{S_t}{\beta(t)}\right) = \frac{1}{\beta(t)} \sigma_t S_t \left[\frac{\mu_t - r_t}{\sigma_t} dt + dW_t\right]$$

$$= \frac{1}{\beta(t)} \sigma_t S_t \left(\gamma_t\, dt + dW_t\right) \text{ where } \gamma_t = \frac{\mu_t - r_t}{\sigma_t}.$$

Furthermore, $d\left(\frac{Z_t}{\beta(t)}\right) = \Delta(t)\, d\left(\frac{S_t}{\beta(t)}\right)$

1 $$= \frac{\Delta(t)}{\beta(t)} \sigma_t S_t \left(\gamma_t\, dt + dW_t\right). \quad (**)$$

With Girsanov's thm, consider the change of measure from P to Q under which

$$W_t^Q = \int_0^t \gamma_u\, du + W_t.$$

From $(**)$, $d\left(\frac{Z_t}{\beta(t)}\right) = \frac{\Delta(t)}{\beta(t)} \sigma_t S_t\, dW_t^Q.$

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Since $\frac{Z_t}{\beta(t)}$ is driftless under Q, it is a martingale under Q.

PROBLEM # 3 (a) [4 PTS] We are given $dr_t = (\alpha - \beta r_t)dt + \sigma dW_t$. In terms of the representation $dr_t = a(b - r_t)\, dt + \sigma dW_t$, we have

$a = \beta$ and $b = \alpha/\beta$ since

$$dr_t = \beta\left(\frac{\alpha}{\beta} - r_t\right) dt + \sigma dW_t.$$

From the representation $dr_t = (\alpha - \beta r_t)dt + \sigma dW_t$, it is clear that as $\beta \to 0$, we obtain the Ho-Lee model
$dr_t = \alpha dt + \sigma dW_t$.

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Previously, in terms of the parameters a, b and σ, we have $B(t,T) = \exp\left(-A(t,T)r_t + D(t,T)\right)$, where $A(t,T) = \dfrac{1-e^{-a(T-t)}}{a}$ and

$$D(t,T) = \left(b - \frac{\sigma^2}{2a^2}\right)\left[A(t,T) - (T-t)\right] - \frac{\sigma^2 A(t,T)^2}{4a}.$$

So, in terms of the parameters α, β and σ, we have

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$$A(t,T) = \frac{1-e^{-\beta(T-t)}}{\beta} \quad \text{and}$$

$$D(t,T) = -\frac{\alpha}{\beta}(T-t) + \frac{\alpha}{\beta^2}\left(1-e^{-\beta(T-t)}\right) + \frac{\sigma^2}{2\beta^2}(T-t) + \frac{\sigma^2}{4\beta^3}\left(1-e^{-2\beta(T-t)}\right) - \frac{\sigma^2}{\beta^3}\left(1-e^{-\beta(T-t)}\right).$$

Hence, $\displaystyle\lim_{\beta\to 0} A(t,T) = \lim_{\beta \to 0} \frac{(T-t)e^{-\beta(T-t)}}{1}$ by L'Hôpital's Rule

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$= (T-t)$.

$$\lim_{\beta \to 0} D(t,T) = -\frac{1}{2}\alpha(T-t)^2 + 1/6\, \sigma^2 (T-t)^3$$

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by applying L'Hôpital's Rule 3 times. DETAILS needed here.

0.5 Finally, $B(t,T) = \exp\left(-(T-t)r_t - 1/2\, \alpha (T-t)^2 + 1/6\, \sigma^2 (T-t)^3\right)$.

(b) 4 PTS For the Ho-Lee model, $r_t = r_0 + \alpha t + \sigma W_t$.

Hence, $r_u = r_t + \alpha(u-t) + \sigma(W_u - W_t)$, $u \geq t$. (*)

$$B(t,T) = E\left[\exp\left(-\int_t^T r_u(r_t)\, du\right)\right]$$ since r_t is Markov.

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$$= \exp\left\{ E\left[-\int_t^T r_u(r_t)\, du\right] + 1/2\, Var\left[-\int_t^T r_u(r_t) du\right]\right\}$$

Since from (*) r_u is normal and $\int_t^T r_u(r_t)\, du$ is Gaussian.

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$$E\left[-\int_t^T r_u(r_t)\, du\right] = -\int_t^T E[r_u(r_t)]\, du$$

$$= -\int_t^T \left(r_t + \alpha(u-t)\, du\right) = -r_t(T-t) - 1/2\, \alpha (T-t)^2.$$

Consider $Var\left[-\int_0^t r_u\, du\right]$.

$$Var\left[-\int_0^t r_u\, du\right] = E\left[\left(-\int_0^t r_u\, du - E\left[-\int_0^t r_u du\right]\right)^2\right]$$

$$= E\left[\left(-\int_0^t (r_u - E[r_u])\, du\right)^2\right]$$

$$= E\left[\left(\int_0^t (r_u - E[r_u])\, du\right)\left(\int_0^t (r_s - E[r_s])\, ds\right)\right]$$

$$= \int_0^t \int_0^t E\left[(r_u - E[r_u])(r_s - E[r_s])\right] du\, ds$$

$$= \int_0^t \int_0^t E\left[(\sigma W_u)(\sigma W_s)\right] du\, ds$$

If you cite instead RS Mamon (2004), JAMDS paper, that's fine, but you need to outline the steps

$$= \int_0^t \int_0^t \sigma^2 (u \wedge s)\, du\, ds = \int_0^t \int_0^s \sigma^2 u\, du\, ds + \int_0^t \int_0^u \sigma^2 s\, ds\, du$$

$$= 2\sigma^2 \int_0^t \int_0^u s\, ds\, du$$

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$$= 2\sigma^2 \int_0^t \left[\left(\frac{1}{2} s^2\right)\right]_0^u du = 2\sigma^2 \cdot \frac{1}{2} \int_0^t u^2 du$$

$$= \frac{1}{3}\sigma^2 t^3.$$

Consequently, $\mathrm{Var}\left[-\int_t^T r_u\, du \,\middle|\, r_t\right]$

$$= \frac{1}{3}\sigma^2 (T-t).$$

0.5 Hence, $B(t,T) = \exp\left(-r_t(T-t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\right).$

$\boxed{4\text{ PTS}}$

PROBLEM #4 Write $U := x+y+z$ and $V := y^2 - xz$

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$$\frac{\partial U}{\partial x} = 1, \quad \frac{\partial U}{\partial y} = 1, \quad \frac{\partial U}{\partial z} = 1,$$ all 2nd-order partial derivatives are 0.

$$\frac{\partial V}{\partial x} = -z, \quad \frac{\partial V}{\partial y} = 2y, \quad \frac{\partial V}{\partial z} = -x$$

$$\frac{\partial^2 V}{\partial x^2} = 0, \quad \frac{\partial^2 V}{\partial y^2} = 2, \quad \frac{\partial^2 V}{\partial z^2} = 0$$ Also, cross-product terms are 0.

$X_t^1 = U, \quad X_t^2 = V$

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$$dX_t = \begin{bmatrix} dX_t^1 \\ dX_t^2 \end{bmatrix} = \begin{bmatrix} dU(W_t^1, W_t^2, W_t^3) \\ dV(W_t^1, W_t^2, W_t^3) \end{bmatrix}$$

By Itô's lemma,

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$$dU_t = \frac{\partial U}{\partial x} dW_t^1 + \frac{\partial U}{\partial y} dW_t^2 + \frac{\partial U}{\partial z} dW_t^3 + \frac{1}{2}\begin{bmatrix} dW_t^1 & dW_t^2 & dW_t^3 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial z^2} \\ \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial z^2} \\ \frac{\partial^2 U}{\partial x \partial x} & \frac{\partial^2 U}{\partial y \partial z} & \frac{\partial^2 U}{\partial z^2} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{bmatrix}$$

Similar dynamics for V_t (i.e., dV_t) can be obtained.

Thus,

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$$dX_t^1 = 1 \cdot dW_t^1 + 1 \cdot dW_t^2 + 1 \cdot dW_t^3 + 0$$

$$dX_t^2 = -W_t^3 dW_t^1 + 2W_t^2 dW_t^2 - W_t^1 dW_t^3 + \frac{1}{2}\begin{bmatrix} dW_t^1 & dW_t^2 & dW_t^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{bmatrix}$$

$$= -W_t^3 dW_t^1 + 2W_t^2 dW_t^2 - W_t^1 dW_t^3 + 1/2 \cdot 2 \, (dW_t^2)^2$$

since $dW_t^i \cdot dW_t^j = 0$ for $i \neq j$ by independence.

So,

$$dX_t = \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \begin{bmatrix} 1 & 1 & 1 \\ -W_t^3 & 2W_t^2 & -W_t^1 \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{bmatrix}$$

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