FM 9561B – Assignment No. 3 Winter 2014

$\boxed{\text{5 POINTS}}$ <u>Outline of Solutions</u>

<u>Problem 1:</u> Consider the SDE $dr_t = \mu(t, r_t)\,dt + \sigma(t, r_t)\,dW_t$

where W_t is a BM under a risk-neutral measure Q. Suppose that the bond price $B(t,T)$ has the exponential affine form

$$B(t,T) = \exp\big(A(t,T) - C(t,T)\,r_t\big).$$

NOTE: Students need to provide detailed calculations or reference properly relevant lecture items.

By the application of Ito's lemma (note that this was done in the lecture when we discussed the Hull-White model), we have

$$dB(t,T) = B(t,T)\Big[\Big(\frac{\partial A}{\partial t} - \frac{\partial C}{\partial t} r_t - C\mu(t,r_t) + \frac{1}{2} C \sigma^2(t,r_t)\Big)dt - C\sigma(t,r_t)\,dW_t\Big] \quad (1)$$

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But, we also know that under a risk-neutral measure Q (again, from the lecture), the bond price has dynamics

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$$dB(t,T) = B(t,T)\big[r_t\,dt + \xi(t,T,r_t)\,dW_t\big] \quad (2)$$

where $\xi(t,T,r_t)$ is the volatility of $B(t,T)$. This equality comes from the requirement that all tradable assets must have expected growth at the risk-free rate under Q.

Let $r_t = r$ and write

$$h(t,r) := \frac{\partial}{\partial t} - \frac{\partial C}{\partial t} r - C\mu(t,r) + \frac{1}{2} C\sigma^2(t,r) - r.$$

By matching drifts of (1) and (2), we

see that $h(t,r) = 0 \ \forall t$ and r.
We differentiate $h(t,r)$ w.r.t. r to obtain

$$\frac{\partial^2 h}{\partial r^2} = -C(t,T)\frac{\partial^2 \mu(t,r)}{\partial r^2} + \frac{1}{2}C(t,T)^2\frac{\partial^2(\sigma(t,r)^2)}{\partial r^2} = 0$$

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Consequently,

$$-\frac{\partial^2 \mu(t,r)}{\partial r^2} + \frac{1}{2}C(t,T)\frac{\partial^2(\sigma(t,r)^2)}{\partial r^2} = 0. \quad (3)$$

Since $C(t,T)$ is a function of T as well as t, identity in equation (3) can only hold if both

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$$\frac{\partial^2(\sigma(t,r)^2)}{\partial r^2} = 0 \quad \text{and} \quad \frac{\partial^2 \mu(t,r)}{\partial r^2} = 0. \quad (4)$$

Clearly, from (4) both $\sigma(t,r)^2$ and $\mu(t,r)$ must be affine in r. This

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means that there are some $\alpha(t)$ and $\beta(t)$ s.t $\mu(r_t,t) = \alpha(t) + \beta(t)r_t$ and there are some $\gamma(t)$ and $\delta(t)$ s.t.
$\sigma^2(r_t,t) = \gamma(t)r_t + \delta(t)$ OR

$$\sigma(r_t,t) = \sqrt{\gamma(t)r_t + \delta(t)}.$$

Problem #2:

Ho-Lee model: $dr_t = \theta dt + \sigma dW_t$.

Simulate r over $[0, T]$, with $0 = t_0 < t_1 < \dots < t_N = T$.

Using the Euler's Scheme, we have the discretised version

$$r_{t_{i+1}} - r_{t_i} = \theta(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\, \xi_{t_{i+1}},$$

$\xi_{t_{i+1}} \sim N(0, 1)$ and independent RVs.

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Let

$\Delta t = t_{i+1} - t_i =$ constant.

So,

$$r_{t+\Delta t} = r_t + \theta \Delta t + \sigma\sqrt{\Delta t}\, \xi_t \sim N(r_t + \theta\Delta t, \sigma^2 \Delta t),$$

where $\xi_t \sim N(0, 1)$.

Suppose $\beta = (\theta, \sigma)$ is the parameter set that determines the model.

The likelihood function $L(\beta)$ is

$$L(\beta) = (2\pi\sigma^2\Delta t)^{-N/2} \exp\left(-\frac{1}{2}\sum_{i=0}^{N-1}\left(\frac{r_{t_{i+1}} - (r_{t_i} + \theta\Delta t)}{\sigma\sqrt{\Delta t}}\right)^2\right).$$

Set $\frac{\partial \ln L(\beta)}{\partial \theta} = 0$ to solve for $\hat{\theta}$ and $\hat{\sigma}$.

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$$\ln L(\beta) = -\frac{N}{2}\ln(2\pi\sigma^2\Delta t) - \frac{1}{2}\sum_{i=0}^{N-1}\left(\frac{r_{t_{i+1}} - (r_{t_i} + \theta\Delta t)}{\sigma\sqrt{\Delta t}}\right)^2$$

$$= -\frac{N}{2}\ln(2\pi\Delta t) - N\ln\sigma - \frac{1}{2\sigma^2\Delta t}\sum_{i=0}^{N-1}\left(r_{t_{i+1}} - r_{t_i} - \theta\Delta t\right)^2.$$

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$$\frac{\partial}{\partial\theta}\ln L(\phi) = -\frac{1}{\gamma}\frac{1}{\sigma^2\Delta t}\sum_{i=0}^{N-1}(-\gamma\Delta t)\left(r_{t_{i+1}}-r_{t_i}-\theta\Delta t\right)$$

$$= \frac{1}{\sigma^2}\sum_{i=0}^{N-1}\left(r_{t_{i+1}}-r_{t_i}-\theta\Delta t\right) = 0. \quad (1)$$

$$\frac{\partial}{\partial\sigma}\ln L(\phi) = -\frac{N}{\sigma}+\frac{1}{\sigma^3}\sum_{i=0}^{N-1}\left(r_{t_{i+1}}-r_{t_i}-\theta\Delta t\right)^2 = 0 \quad (2)$$

From equation (1), we get

$$\sum_{i=0}^{N-1}\left(r_{t_{i+1}}-r_{t_i}\right) = N\theta\Delta t \quad (3)$$

0.5 and from (2), we have

$$\sum_{i=0}^{N-1}\left(r_{t_{i+1}}-r_{t_i}-\theta\Delta t\right)^2 = N\sigma^2\Delta t \quad (4)$$

Solving (3)-(4) gives

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$$\hat{\theta} = \frac{1}{N\Delta t}\sum_{i=0}^{N-1}\left(r_{t_{i+1}}-r_{t_i}\right) = \frac{r_T-r_0}{T} \quad \because t_n = T \text{ \& } t_0 = 0$$

$$\hat{\sigma} = \left[\frac{1}{N\Delta t}\sum_{i=0}^{N-1}\left(r_{t_{i+1}}-r_{t_i}-\theta\Delta t\right)^2\right]^{1/2}.$$

Note: $\frac{T}{N} = \Delta t$

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$$= \left[\frac{1}{T}\sum_{i=0}^{N-1}\left(r_{t_{i+1}}-r_{t_i}-\frac{r_T-r_0}{N}\right)^2\right]^{1/2}.$$

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Problem #3

(b) $\sigma(t,T) = v e^{-\varsigma(T-t)}$, $\sigma>0$. The corresponding HJM equation is $df(t,T) = v e^{-\varsigma(T-t)} dW_t$

$$+ \left(v e^{-\varsigma(T-t)} \int_t^T v e^{-\varsigma(T-t)} du \right) dt$$

$$= v e^{-\varsigma(T-t)} dW_t + v e^{-\varsigma(T-t)} \frac{v}{\varsigma}\left(1 - e^{-\varsigma(T-t)}\right) dt$$

$$= v e^{-\varsigma(T-t)} dW_t + \frac{v^2}{\varsigma}\left(e^{-\varsigma(T-t)} - e^{-2\varsigma(T-t)}\right) dt$$

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$$\therefore f(t,T) = f(0,T) + v\int_0^t e^{-\varsigma(T-u)} dW_u$$

$$+ \frac{v^2}{2\varsigma}\left[\left(1 - e^{-\varsigma T}\right)^2 - \left(1 - e^{-\varsigma(T-t)}\right)^2 \right]$$

Using the fact that $r_t = f(t,t)$, we have

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$$r_t = f(0,t) + \underbrace{v\int_0^t e^{-\varsigma(t-u)} dW_u}_{=Y_t} + \frac{v^2}{2\varsigma}\left(1 - e^{-\varsigma t}\right)^2. \quad (A)$$

Consider $Y_t := v\int_0^t e^{-\varsigma(t-u)} dW_u$

$$= \left(v e^{-\varsigma t}\right)\left(\int_0^t e^{\varsigma u} dW_u\right).$$

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Hence, $dY_t = v\, dW_t - v\varsigma \int_0^t e^{-\varsigma(t-u)} dW_u\, dt$ by the Chain Rule/ Itô's lemma

OR $\boxed{dY_t = v\, dW_t - \varsigma Y_t\, dt}$ (B)

Also, from (A),

$$Y_t = r_t - f(0,t) - \frac{v^2}{2\varsigma}\left(1 - e^{-\varsigma t}\right)^2. \quad (C)$$

Using (A), (B) and (C), we get

$$dr_t = \left(\frac{\partial}{\partial T} f(0,T) \Big|_{T=t} \right) dt$$

$$+ \sigma dW_t - \varsigma r_t \, dt$$

$$+ \frac{\sigma^2}{\varsigma} e^{-\varsigma t} \left(1 - e^{-\varsigma t}\right) dt .$$

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where $\beta(t) = f(0,t) + \frac{1}{\varsigma} \frac{\partial}{\partial T} f(0,T) \Big|_{T=t}$

$$+ \frac{\sigma^2}{2\varsigma^2} \left(1 - e^{-2\varsigma t}\right).$$

Clearly, the specification in (D) is the Hull-White model for the short-rate process r_t.

1 PTS (a) The choice of $\sigma(t,T) = \sigma e^{-\varsigma(T-t)}$ is due to empirical evidence that the forward rate volatility has an exponentially decaying behaviour versus maturity $(T-t)$.

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Problem 4:

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The value of a consol G_t is given by

1 $G(t) = \int_t^\infty c \cdot B(t,s)\, ds$. We employ the Leibniz rule for differentiation to obtain $dC(t)$.

Recall $\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,y)\, dy\right] = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,y)\, dy + f(x, b(x)) \frac{d}{dx} b(x)$

$- f(x, a(x)) \frac{d}{dx} a(x).$

In our case, $a(t) = t$ and $b(t) = \infty$.

1 $\frac{dG(t)}{dt} = c\left(\int_t^\infty \frac{d}{dt} B(t,s)\, ds + 0 - \cancel{B(t,t)}^{1}\right)$

$dC(t) = c\left(\int_t^\infty dB(t,s)\, ds - dt\right)$.

Risk-neutral dynamics of bond price.

Given: $dB(t,s) = B(t,s) r_t\, dt + B(t,s)\sigma(t,s)\, dW_t$

1 $= c \int_t^\infty [B(t,s) r_t\, dt + B(t,s)\sigma(t,s)\, dW_t]\, ds$

$- c\,dt$.

Hence, the volatility component of $C(t)$ is given by

1 $c \int_t^\infty B(t,s)\sigma(t,s)\, ds.$