FM9561B – 27 – 30 January 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. Cross variations of BMs: Since each component W_t^j is a one-dimensional BM, we have $dW_t^i dW_t^i = dt$ and if $i \neq j$ then $dW_t^i dW_t^j = 0$. Note that a sketch of the proof of these results was presented in the lecture.

2. Multi-dimensional Itô formula: We write the Itô formula for two processes driven by a 2-dimensional BM. The formula generalises to any number of processes driven by a BM of any number of dimensions.

   Consider the process $\mathbf{Z}_t = \begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \mathbf{Z}_0 + \int_0^t \boldsymbol{\gamma}_s ds + \int_0^t \mathbf{K}_s d\mathbf{W}_s$. Here, $\mathbf{Z}_0 = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$, $\boldsymbol{\gamma}_s = \begin{bmatrix} \alpha_s \\ \beta_s \end{bmatrix}$, $\mathbf{K}_s = \begin{bmatrix} K_s^{11} & K_s^{12} \\ K_s^{21} & K_s^{22} \end{bmatrix}$ and $d\mathbf{W}_s = \begin{bmatrix} dW_s^1 \\ dW_s^2 \end{bmatrix}$.
   Remark: Such processes, consisting of a non-random initial condition, plus a Riemann integral, plus one or more Itô integrals, are called **semi-martingales**.

   The integrands α_s, β_s and K_s^{ij} can be any adapted processes.

   Let $f(t, x, y)$ be a function of 3 variables and suppose X_t and Y_t are semi-martingales with dynamics given above. Then,

$$\begin{aligned} f(t, X_t, Y_t) &= f(0, X_0, Y_0) + \int_0^t \left(\frac{\partial f}{\partial s} + \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} + \frac{1}{2} \left[(K^{11})^2 + (K^{12})^2 \right] \frac{\partial^2 f}{\partial x^2} \right. \\ &\quad \left. + (K^{11} K^{21} + K^{12} K^{22}) \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2} \left[(K^{21})^2 + (K^{22})^2 \right] \frac{\partial^2 f}{\partial y^2} \right) ds \\ &\quad + \int_0^t \left(K^{11} \frac{\partial f}{\partial x} + K^{21} \frac{\partial f}{\partial y} \right) dW^1 + \int_0^t \left(K^{12} \frac{\partial f}{\partial x} + K^{22} \frac{\partial f}{\partial y} \right) dW^2. \end{aligned}$$

3. Stochastic Differential Equation (SDE): An SDE is an expression of the form $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$. Here, W is a standard BM on $(\Omega, \mathcal{F}, P)$ and $X_0 \in I\!R$ is given. A process X is a **solution** of the SDE if

$$X_t = X_0 + \int_0^t \mu(s, X_x)ds + \int_0^t \sigma(s, X_s)dW_s \;\; \forall t \geq 0, \;\text{ and } \forall \omega \in \Omega, \text{ a.e.}$$

A solution to the SDE above **exists** if the drift and volatility components satisfy the **Lipschitz conditions**. These conditions were discussed in the lecture. The solution is also adapted to $\{\mathcal{F}_t\}$.

4. Girsanov theorem.
(i) Suppose γ is an $\mathcal{F}_t$−adapted process. Consider $\Lambda_t := \exp\left(-\int_0^t \gamma_s dW_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right)$ and $\int_0^t \gamma_s dW_s$ is a stochastic integral. We require $E\left[\int_0^t \gamma_s^2 ds\right] < \infty$. We **proved** that Λ_t is a martingale (using the martingale property of a stochastic integral). A sufficient condition for Λ_t to be a martingale is the so-called **Novikov's condition**; this was also discussed in class.

(ii) Also, $E[\Lambda_t] = 1$ and $\Lambda_t \geq 0$ So, Λ_t can be a candidate for a density.

(iii) We define a new probability measure Q on $\mathcal{F}_T$ via the Radon-Nikodŷm derivative $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = \Lambda_T$. In other words, if $A \in \mathcal{F}_T$ then $Q(A) := \int_A \Lambda_T dP$.

(iv) We note that W is no longer a BM under Q. However, W_t^Q is a BM under Q where $W_t^Q := W_t + \int_0^t \gamma_s ds$.

(v) We proved that when we introduce the measure Q, we remove the mean γt (assuming γ is constant in (iv)) from W_t^Q. The measure Q changes W_t^Q so that it has a zero mean.

5. When we use Girsanov' theorem to change probability measure, *means change but variances do not and hence;* **martingales may be destroyed or created.**

6. Calculation of the Unconditional Expected Value of a RV ϕ under Q but in terms of measure P: For an $\mathcal{F}_T-$measurable ϕ,

$$E^Q[\phi] = E[\Lambda_T \phi].$$

(ii) Calculation of the Conditional Expected Value of a RV ϕ under Q but in terms of measure P: We have here the Bayes' rule for conditional expectation given by

$$E^Q[\phi|\mathcal{F}_s] = \frac{E[\Lambda_T \phi|\mathcal{F}_s]}{E[\Lambda_T|\mathcal{F}_s]}.$$

That is, $E[\Lambda_T \phi|\mathcal{F}_s] = E^Q[\phi|\mathcal{F}_s]E[\Lambda_T|\mathcal{F}_s]$ for $s \leq T$.

7. Using the Bayes' rule, we showed in class that W^Q is a martingale under the measure Q.

8. We considered a market model consisting of (i) a bank account that earns the interest rate following an $\mathcal{F}_t-$adapted process r_t, $0 \leq t \leq T$ and (ii) a risky asset ("stock") S_t, $0 \leq t \leq T$. Assume $dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$ where μ_t and σ_t are adapted to $\mathcal{F}_t$.

9. We examined the dynamics of the wealth process Z of an agent who invests in the riskless asset (bank account) and a risky asset. In particular, $dZ_t = \Delta(t)dS_t + r_t(Z_t - \Delta(t)S_t)dW_t$ where $\Delta(t)$ is the number of shares that the investor needs to buy/hold at time t. The first component of the increment dZ_t is the capital gains from the stock whilst the second term is the interest earnings.

10. Denoting the compounding factor by $\beta(t) := \exp\left(\int_0^t r_u du\right)$ so that $d\beta(t) = r_t\beta(t)dt$, we showed that $d\left(\frac{S_t}{\beta(t)}\right) = \frac{1}{\beta(t)}\sigma_t S_t[\gamma(t)dt + dW_t]$ where $\gamma(t) = \frac{\mu_t - r_t}{\sigma_t}$. We call γ_t as the *market price of risk*. Consequently, by the measure change $W_t^Q := W_t + \int_0^t \gamma(u)du$, we obtain $d\left(\frac{S_t}{\beta(t)}\right) = \frac{1}{\beta(t)}\sigma_t S_t dW_t^Q$. This means that the discounted price process $\widetilde{S}_t := \frac{S_t}{\beta(t)}$ is a martingale. Similarly, it may be shown that the discounted wealth process $\widetilde{Z}_t := \frac{Z_t}{\beta(t)}$ is also a martingale.