FM9561B – 24 – 28 February 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. Using the fact that under the Vasiček model r_t is Markov and $B(t,T,r) := E\left[\exp\left(-\int_t^T r_u(r)du\right)\bigg| r_t = r\right]$ with $r_u = e^{-a(u-t)}\left[r + b(e^{a(u-t)} - 1) + \sigma\int_t^u e^{a(s-t)}dW_s\right]$, we show that $\frac{\partial B}{\partial r} = -A(t,T)B(t,T,r)$. Thus,
$$B(t,T,r) = C(t,T)\exp(-A(t,T)r), \tag{1}$$
where $A(t,T) = \frac{1-e^{-a(T-t)}}{a}$. Our aim is to find the deterministic function $C(t,T)$ (and independent of r) in equation (1).

2. To derive the bond price PDE, we considered
$$E\left[\exp\left(-\int_0^T r_u du\right)\bigg| \mathcal{F}_t\right] := \exp\left(-\int_0^t r_u du\right)B(t,T,r), \tag{2}$$
which is a martingale under the pricing measure. We shall apply Itō's lemma to equation (2) and note that the du terms must be identical to zero since this is a martingale. This will give rise to the PDE satisfied by the bond price.

   Consequently, the bond price PDE is derived as
$$-r_t B(t,T,r) + \frac{\partial}{\partial t}B(t,T,r) + \frac{\partial}{\partial r}B(t,T,r)(a(b-r)) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2}B(t,T,r) = 0$$
with $B(T,T,r) = 1$ for all r.

3. Consider the bond price PDE

$$\begin{aligned}-r_t B(t,T,r) &+ \frac{\partial}{\partial t}B(t,T,r) + \frac{\partial}{\partial r}B(t,T,r)(a(b-r)) \\ &+ \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2}B(t,T,r) = 0 \end{aligned} \tag{3}$$

with $B(T,T,r) = 1$ for all r.

From #1, under the Vasiček model, the form of the bond price $B(t,T,r)$ as in equation (1), is given by

$$B(t,T,r) = C(t,T)\exp(-A(t,T)r),$$

where $A(t,T) = \dfrac{1-e^{-a(T-t)}}{a}$ and $C(t,T)$ is a deterministic function.

Taking the partial derivatives of equation (1) and substituting these into equation (3), we obtain the ODE

$$-rC + \frac{\partial C}{\partial t} - C\frac{\partial A}{\partial t}r - AC(a(b-r)) + \frac{\sigma^2}{2}A^2C = 0.$$

Now, $B(t,T,0) = C(t,T)$ and by putting $r = 0$, we get $\dfrac{\partial C}{\partial t} - abAC + \dfrac{\sigma^2}{2}A^2C = 0$ with $C(T,T) = 1$. Solving this ODE, we get the bond price $B(t,T,r) = C(t,T)\exp(-A(t,T)r) = \exp(-A(t,T)r + D(t,T))$ where the function $A(t,T)$ is defined in #1 above whilst

$$\begin{aligned}C(t,T) &= \exp\Big[-b(T-t) + \frac{b}{a}(1-e^{-a(T-t)}) + \frac{\sigma^2}{2a^2}(T-t) \\ &\quad + \frac{\sigma^2}{4a^3}(1-e^{-2a(T-t)}) - \frac{\sigma^2}{a^3}(1-e^{-a(T-t)})\Big]\end{aligned}$$

and $D(t,T) := \log C(t,T)$.

4. Hull and White (1990) extended the Vasiček model by making the parameters time-dependent. In particular, the short rate process r_t

follows the dynamics $dr_t = (\alpha(t) - \beta(t)r_t)dt + \sigma(t)dW_t$. If we write $b_t := \int_0^t \beta(u)du$ then

$$r_t = e^{-b(t)} \left[r_0 + \int_0^t e^{b(u)}\alpha(u)du + \int_0^t e^{b(u)}\sigma(u)dW_u \right].$$

It may be verified that

$$E\left[\int_0^T r_t dt\right] = \int_0^T e^{-b(t)} \left[r_0 + \int_0^t e^{b(u)}\alpha(u)du \right] dt$$

and

$$\mathrm{Var}\left[\int_0^T r_t dt\right] = \int_0^T e^{2b(u)}\sigma^2(u) \left(\int_0^T e^{-b(s)}ds \right)^2 du.$$

5. Using the results for the mean and variance above, analytic expression for $B(0, T, r)$ is obtained (which were shown in class). Similarly, the zero-coupon bond price at time $t < T$ is

$$B(t, T, r) = E\left[\exp\left(-\int_t^T r_u(r)du \right) \middle| \mathcal{F}_t \right] = \exp(-A(t, T)r - C(t, T))$$

where

$$A(t, T) = e^{b(t)} \int_t^T e^{-b(u)}du = e^{b(t)}\gamma(t, T), \quad \gamma(t, T) = \int_t^T e^{-b(u)}du$$

and

$$C(t, T) = \int_t^T \left(e^{b(u)}\alpha(u)\gamma(t, u) - \frac{1}{2}e^{2b(u)}\sigma^2(u)\gamma(t, u) \right) du.$$

The bond price dynamics under the Hull-White model can also be obtained by finding $dB(t, T)$ through the application of Itō's lemma to the explicit solution $B(t, T, r)$ given above.

6. European call on a zero-coupon bond: We considered the price at time t of a call with exercise date T_1 on a bond with maturity T_2 where $t < T_1 < T_2$. Denote by $X := r(T_1)$ and $Y := \int_0^{T_1} r(u)du$. The means and variances of the random variables X and Y can be calculated in a straightforward manner. Suppose $E[r(T_1)] = m_1$ and $\mathrm{Var}[r(T_1)] = \sigma_1^2$, $E\left[\int_0^{T_1} r_u du\right] = m_2$ and $\mathrm{Var}\left[\int_0^{T_1} r(u)du\right] = \sigma_2^2$. Also, write $\rho := \dfrac{\mathrm{Cov}\left[r(T_1), \int_0^{T_1} r(u)du\right]}{\sigma_1\sigma_2}$ and let $f(x,y)$ be the Gaussian density of the bivariate random variable (X,Y). Then, the price at time 0 of a European call on the bond $B(T_1,T_2)$ is given by

$$\begin{aligned} c(0) &= E\left[\exp\left(-\int_0^{T_1} r(u)du\right)(B(T_1,T_2)-K)^+\right] \\ &= E\left[\exp\left(-\int_0^{T_1} r(u)du\right)(\exp(-A(T_1,T_2)r_{T_1} - C(T_1,T_2)) - K)^+\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-y}\left(\exp(-A(T_1,T_2)x - C(T_1,T_2)) - K\right)^+ f(x,y)dxdy. \end{aligned}$$

7. In the Vasiček and Hull-White models, the short rate r_t is Gaussian and so it can be negative with positive probability. Cox, Ingersoll and Ross (CIR, 1985) proposed a model where the short term rate does not take negative values. Consider n Ornstein-Uhlenbeck processes $X^1, \ldots, X^n$ where each X^i has dynamics $X_t^i = -\frac{\alpha}{2}X_t^i dt + \frac{\sigma}{2}dW_t^i$ and X_0^i is known. So, $X_t^i = e^{-\frac{\alpha}{2}t}\left(X_0^i + \frac{\sigma}{2}\int_0^t e^{\frac{\alpha}{2}u}dW_u^i\right)$ and $(W^1, \ldots, W^n)^\top := W$ is an $n-$dimensional BM on $(\Omega, \mathcal{F}, P)$; $\top$ denotes the transpose of a vector. We define the process $r_t := (X_t^1)^2 + (X_t^2)^2 + \ldots + (X_t^n)^2$. Our two main objectives are to find the dynamics of r_t and the form of the bond price under this short term rate specification.