# FM9561B – 03 – 07 February 2014

## SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. We define a *risk-neutral measure* as any probability measure equivalent to the market measure $P$, which makes all discounted asset prices martingales.

2. We also showed that under $Q$, the risky asset $S$ and the bank account have the same rate of return.

3. A *contingent claim* $h$ in a contract with exercise date $T$ is an $\mathcal{F}_T-$measurable RV. Some examples were given in class.

4. An *arbitrage* is a strategy such that the wealth process $Z_t$ satisfies $Z(t,\omega) = x \leq 0$ a.s., $Z(T,\omega) \geq 0$ a.s. and $P(Z(T,\omega) > 0) > 0$ for $t < T$.

5. We considered the problem: what should we pay at time 0 for a contingent claim $h$ payable at time $T$? Here, we assume that there is a hedging process $\Delta(t)$ and a corresponding wealth process $Z_t$ such that $Z_T = h(\omega)$. It will be shown that the correct price at time 0 for the contingent claim $h$ is given by $E^Q\left[\dfrac{h}{\beta(T)}\right] = Z_0$.

6. The pricing formula $E^Q\left[\frac{h}{\beta(T)}\right] = Z_0$ depends on the existence of the hedging rule $\Delta(t)$; this existence is guaranteed by the Martingale Representation Theorem, which shall be discussed after the Reading Week.

7. **Martingale Representation Theorem (MRT):** Let $M$ be an $(\mathcal{F}_t, P)$ martingale. Then there is a process $H_t$ such that $M_t = M_0 + \int_0^t H_u dW_u$.

8. It was shown that the dynamics of $d\widetilde{Z}_t = d\left(\frac{Z_t}{\beta(t)}\right) = \frac{\Delta(t)}{\beta(t)}\sigma_t S_t dW_t^Q$ and hence $\widetilde{Z}_t$ is a martingale under the measure $Q$. It is therefore clear from the statement of the MRT that $\Delta(t) = \frac{\beta(t)H_t}{\sigma_t S_t}$. We emphasise that the MRT guarantees the existence of a hedging portfolio. It does not however tell us how to compute it. The important point to remember is that it does justify the risk-neutral pricing.

9. We specify a portfolio in terms of $(\theta_t^0, \theta_t^1)$ where $\theta_t^i$ :=no. of units of $S_t^i$ held at time $t$. The vector $(\theta_t^0, \theta_t^1)$ can be viewed as a portfolio process or hedging strategy. We previously worked with the wealth process $Z_t$, $\theta_t^1 = \Delta(t)$, and $S_t^0 = \beta(t) = \exp\left(\int_0^t r_u du\right)$.

10. The strategy $(\theta_t^0, \theta_t^1)$ is **self-financing** if the only changes in the corresponding wealth process $Z_t$ come from changes in $S_t^0$ and $S_t^1$, that is, $dZ_t = \theta_t^0 dS_t^0 + \theta_t^1 dS_t^1$, where $S_t^0$ and $S_t^1$ refer to the risky and riskless assets, respectively.

11. In terms of the new $(\theta_{t^1}, \theta_{t^2})$ notation, the discounted wealth process

under the risk-neutral measure $Q$ has dynamics given by $d\left(\frac{Z_t}{S_t^0}\right) = \theta_t^1 \sigma_t \frac{S_t^1}{S_t^0} dW_t^Q$ and hence it is a martingale.

From the MRT, the hedging strategy is justified by the existence of the process $H_u$ such that $\frac{Z_t}{S_t^0} := M_t = M_0 + \int_0^t H_u dW_u^Q$. By matching volatility, we find that $\theta_t^1 = \frac{H_t S_t^0}{S_t^1 \sigma_t}$ and $\theta_t^0 = \frac{Z_t}{S_t^0} - \theta_t^1 \frac{S_t^1}{S_t^0}$.

12. So, for any contingent claim $h$ with exercise time $T$, $\frac{Z_T}{S_T^0} = E^Q\left[\frac{h}{S_T^0}\middle| \mathcal{F}_T\right]$. Also, $Z_T = \theta_T^0 S_T^0 + \theta_T^1 S_T^1 = h$; such property of $Z$ is called **self-replicating**. This implies that $Z_t$ is the rational price at time $t$ for the claim $h$ and $Z_t = E^Q\left[\exp\left(-\int_t^T r_u du\right) h \middle| \mathcal{F}_t\right]$.

13. A zero-coupon bond with maturity $T$ is a claim which pays \$1 at time $T$, i.e., $h(\omega) = 1$. From the risk-neutral valuation, the rational price at time $t$, $0 \leq t \leq T$ for such a default-free zero is $B(t,T) := E^Q\left[\exp\left(-\int_t^T r_u du\right)\middle| \mathcal{F}_t\right]$.

14. A **numéraire** is a positive process. It was demonstrated in the lecture that other assets can be priced in terms of the numéraire. Certain examples were presented in class.