FM9561B – 03 – 07 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. CIR model: The dynamics of the short rate r_t follow the SDE specification $dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$ for constants a, b and σ. Such an r is Markov and a Bessel process. Note that the short rate cannot go negative.

2. We invoked Lévy's theorem in the discussion of the CIR model. Furthermore, we gave conditions (courtesy of Révuz and Yor) on how to choose parameters appropriately to ensure that r does not attain the level 0.

3. It was shown that the price of a zero-coupon bond under the CIR model has an exponential affine form. The Bayes' theorem for conditional expectation was employed to arrive at this result (exponential affine form). One can make use of the distributional property of r_t or the PDE approach to calculate the bond price. Our approach in obtaining the bond price is the forward measure approach.

4. As mentioned in #3, we demonstrated in the lecture that the price of a zero-coupon bond under the CIR model ($dr_t = \beta(\alpha - r_t)dt + \sigma\sqrt{r_t}dW_t$) has an exponential affine form. One can make use of the distributional property of r_t or the PDE approach to calculate the bond price. The closed-form solution of the CIR bond price was given in class. Our approach makes use of the forward measure, the construction of which through the appropriate Radon-Nikodým derivative by specifying the

γ_t-process, was given in class. Under the forward measure, we ended up solving a second-order differential equation.

Under the CIR model, the bond price has the form

$$B(t,T) = \exp(-I(t,T)r + A(t,T)),$$

where $I(t,T)$ and $A(t,T)$ are deterministic functions. The function I satisfies a second-order ordinary differential equation. In particular, the solutions are:

$$I(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{\beta - \gamma + (\beta+\gamma)e^{\gamma(T-t)}}$$

where $\gamma = \sqrt{\beta^2 + 2\sigma^2}$ and

$$A(t,T) = \frac{2\alpha\beta}{\sigma^2} \log\left(\frac{2\gamma e^{(\beta+\gamma)(T-t)/2}}{(\beta+\gamma)e^{\gamma(T-t)} + \gamma - \beta}\right).$$

The relation between the Brownian motion under the risk-neutral measure and the Brownian motion under the forward measure was also presented.