FM9561B – 06 – 10 January 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. Probability space

2. Random variable

3. Stochastic process and its sample path

4. Filtration

5. Variation of a function g on $[0, T]$, which measures the total amount of up and down motion of the paths.

6. Bounded variation: We note that functions of bounded variation on $[0, T]$ are those functions for which the total variation on $[0, T]$ is finite.

7. Quadratic variation of a function g denoted by $\mathrm{QV}_g([0, T])$: We proved the result that if g is differentiable then $\mathrm{QV}_g([0, T])=0$.

8. The contrapositive of the result highlighted in #7 says that "If $\text{QV}_g([0, T])$ is not zero then the function g is not differentiable". This result was employed to show that the sample path of a Brownian motion (BM) is not differentiable.

9. Definition of a BM: Given a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, the stochastic process $W(t, \omega)$: $[0, \infty] \times\Omega \to I\!R$ is a Brownian motion if it satisfies the following properties: (i) $W_0 = 0$, technically $P(\omega : W(0, \omega) = 0) = 1$; (ii) $W_t(\omega)$ is a continuous function of t; (iii) if $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = t$ then the increments $W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent normals with

$$E\left[W_{t_{i+1}} - W_{t_i}\right] = 0 \quad \text{and} \quad E\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2\right] = t_{i+1} - t_i.$$

10. A brief historical development of BM was re-visited:
• In 1828-29, *Robert Brown* observed pollen particles under a microscope that were in constant irregular motion. This phenomenon (called Brownian motion) was named after him.
• In 1900, *Louis Bachelier* used BM for the first time to model stock prices and analysed its implication in option pricing. This is contained in his PhD thesis "Theorie de la Spéculation".
• *Albert Einstein* employed BM in 1905 as a model for particles in suspension and utilised it to calculate Avogadro's number.
• In 1923, *Norbert Wiener* defined and constructed BM rigorously.
• *Paul Samuelson* won the Nobel Prize in Economic Sciences in 1970 for popularising quantitative methods in economics and the use of BM.

11. We showed (heuristically) that the quadratic variation of a Brownian motion on $[0, T]$, denoted by $\langle W \rangle (\omega, T)$, is given by $\langle W \rangle (\omega, T) = T$ *a.s.* Thus, BM's sample paths are not differentiable from #8.

12. The distinction between Riemann integral or Riemann-Stieltjes integral (i.e., integral with respect to a differentiable and monotonic function) and Itô integral was discussed. In particular, we wish to construct integrals of the form $I(t) = \int_0^t g(s,\omega)dW(s,\omega)$. The integrator is a Brownian motion with associated filtration $\{\mathcal{F}_t\}$. The integrand is $g(s)$, $s \geq 0$, where
(i) $g(s)$ is $\mathcal{F}_s-$measurable $\forall s$ and
(ii) g is square-integrable, i.e., $E\left[\int_0^t g^2(s)ds\right] < \infty$, $\forall t \geq 0$.

13. We constructed the Itô integral of an elementary integrand. We defined an elementary/simple process. If $g(t)$ is constant on each subinterval $[t_i, t_{i+1})$ given a partition of $[0, T]$, then g is called an *elementary process.*

14. The Itô integral of an elementary integrand was defined. If $g(t)$ is constant on each subinterval $[t_i, t_{i+1})$ given a partition of $[0, T]$, then g is called an *elementary process.* We have seen that the Itô integral can be viewed as the "gain from trading" at time t.

15. Properties of the Itô integral of elementary processes: Adaptedness, linearity, martingale property and Itô isometry.
Adaptedness: For every t, $I(t)$ is $\mathcal{F}_t-$measurable.
Linearity: If $I(t) = \int_0^t g(s)dW_s$ and $J(t) = \int_0^t h(s)dW_s$ then for constants α and β, $\alpha I(t) \pm \beta J(t) = \alpha \int_0^t g(s)dW_s \pm \beta \int_0^t h(s)dW_s$.

The last remaining properties (martingale property and Itô isometry) will be demonstrated in the succeeding lectures.

16. We established in class that, in general, if g is an elementary process and $t_i \leq t \leq t_{i+1}$ then $I(t)$ takes the form

$$I(t) = \sum_{j=0}^{i-1} g(t_j) \left(W_{t_{j+1}} - W_{t_i} \right) + g(t_i)(W_t - W_{t_i}).$$