FM9561B – 10 – 14 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. Fundamental Theorem of Asset Pricing due to Harrison and Pliska (1979, 1981) (within the context of term structure modelling): A term structure model is free of arbitrage if and only if there is a probability measure Q (a risk-neutral measure) on $(\Omega, \mathcal{F})$ with the same null sets as P such that for each $T \in [0, T^*]$ the process $\dfrac{B(t,T)}{\beta(t)}$, $0 \leq t \leq T$ is a martingale under Q. Here, $\beta(t) = \exp\left(\int_0^t r_u du\right)$.

2. It was argued that Q is a risk-neutral measure if and only if the mean rate of return of $B(t,T)$ under Q is the interest rate r_t. If the mean rate of return of $B(t,T)$ under a measure P is not r_t at each time t and for each maturity T, we should change to a measure Q under which the mean rate of return is r_t. If such measure does not exist then the model admits arbitrage.

3. (i) Pricing formula for a zero-coupon bond:
   $B(t,T) = E\left[\exp\left(-\int_t^T r_u du\right)\bigg| \mathcal{F}_t\right], \quad 0 \leq t \leq T.$

   (ii) Pricing formula for a coupon-paying bond: Assume that we have coupon payments p_1, p_2, ..., p_n at times T_1, T_2, ... T_n, respectively. Then the price at time t of a coupon-paying bond is $\sum_{\{t: t<T_k\}} p_k B(t, T_k)$, where $B(t, T_k)$ is the price of a zero-coupon bond at time t with maturity T_k.

(iii) European call option on a zero-coupon bond: Bond matures at time T whilst the option expires at time $T_1 < T$. Price at time t is

$$\beta(t)E\left[\frac{1}{\beta(T)}\left(B(T_1,T)-K\right)^+|\mathcal{F}_t\right], \quad 0 \le t \le T,$$

where K is the strike price.

4. We defined the forward rate in order to motivate the alternative approach in term structure modelling by Heath, Jarrow and Morton (1992). The instantaneous rate for the amount of dollars borrowed at time T, agreed upon time $t \le T$, is the forward rate $f(t,T)$. We showed in class that the bond price $B(t,T)$ and the forward rate $f(t,T)$ are related through the relation

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \quad \text{or} \quad f(t,T) = -\frac{\partial}{\partial T}\log B(t,T).$$

5. For every $T \in [0,T^*]$, we consider the forward rate having dynamics $df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$. Both α and β are measurable in (t,ω).

6. We showed that $r_t = f(t,t)$.

7. With $d\left[-\int_t^T f(t,u)du\right] = f(t,t)dt - \int_t^T df(t,u)du \overset{i.e.}{=} r_t dt - \alpha^*(t,T)dt - \sigma^*(t,T)dW_t$, we aim to determine the bond price dynamics $dB(t,T)$. This will guide us in setting conditions necessary to obtain arbitrage-free bond prices under the HJM framework.

8. Under the HJM approach, it suffices to specify the volatility function $\sigma(t,T)$. Note that the no-arbitrage condition implies $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du$, $0 \le t \le T \le T^*$.

9. If the equation for $\alpha(t,T)$ in #8 does not hold then we need to find a risk-neutral measure via the Girsanov change of measure. If $\widetilde{W}$ is a Brownian motion under a new measure (risk-neutral) then it is related to the original Brownian motion W under P through the equation $\widetilde{W}_t = \int_0^t \theta(u)du + W_t, \quad \Lambda(t) = \exp\left\{-\int_0^t \theta(u)dW_u - \frac{1}{2}\int_0^t \theta(u)^2 du\right\}$ and $\widetilde{P} = \int_A \Lambda(T^*)dP \quad \forall A \in \mathcal{F}_{T^*}$. In this case, $\alpha(t,T) = \sigma(t,T)\sigma^*(t,T) + \sigma(t,T)\theta(t)$.

10. The main result under the HJM framework is the specification of the forward rate dynamics $df(t,T) = \sigma(t,T)\sigma^*(t,T)dt + \sigma(t,T)d\widetilde{W}_t$. Here, the initial forward curve is determined by the market since $f(0,T) = -\frac{\partial}{\partial T}\log B(0,T), \quad 0 \le T \le T^*$.

11. We also defined the the yield rate $Y(t,T)$ satisfying the equation $B(t,T) = \exp(-Y(t,T)(T-t))$.