FM9561B – 13 – 17 January 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. We showed that if $I(t)$ is an Itô integral, then $I(t)$ is a martingale.

2. We also proved the result known as Itô isometry:
$$E\left[I^2(t)\right] = E\left[\left(\int_0^t g(s)dW_s\right)^2\right] = E\left[\int_0^t g^2(s)ds\right].$$

3. We considered the construction of an Itô integral for a general integrand. For a fix $T > 0$, we suppose that g is a process (not necessarily an elementary process) that belong to our class of "suitable" integrands. That is, g satisfies the following:
(i) $g(t)$ is $\mathcal{F}_t$-measurable and
(ii) $E\left[\int_0^T g^2(t)dt\right] < \infty$.
The idea is to use the fact that "there is a sequence of elementary processes $\{g_n\}_{n=1}^{\infty}$ such that $E\left[\int_0^T |g_n(t) - g(t)|^2 dt\right] \to 0$ as $n \to \infty$". We then define
$$\int_0^T g(t)dW_t := \lim_{n\to\infty} \int_0^T g_n(t)dW_t.$$
We showed that such limit exists and there is no difficulty with this approach.

4. Summary of the properties of a **general Itô integral** $I(t)$.
(i) Linearity

(ii) Adaptedness
(iii) $I(t)$ is a martingale.
(iv) $I(t)$ is a continuous function of the upper limit of t.
(v) Itô isometry: $E\left[I^2(t)\right] = E\left[\int_0^t g^2(s)ds\right]$.

5. We examined the Itô integral $\int_0^T W_s dW_s$. We checked that this integral is indeed well-defined. That is, W_s is $\mathcal{F}_s^W-$measurable and $E\left[\int_0^T W_s^2 ds\right] < \infty$. As W is a "suitable" integrand, it makes sense to discuss $\int_0^T W_s dW_s$.

6. By approximation, we proved that

$$\int_0^T W_u dW_u = \lim_{n\to\infty} \sum_{i=0}^{n} W_{\frac{iT}{n}} \left(W_{\frac{(i+1)T}{n}} - W_{\frac{iT}{n}}\right) = \frac{1}{2}W_T^2 - \frac{1}{2}T. \qquad (1)$$

   Remark: The extra $\frac{1}{2}T$ comes from the nonzero quadratic variation of BM W_u. Note that the expectation for both the left hand side and right hand side of equation (1) is equal to 0.

7. Consider $I(t) = \int_0^t g(s)dW_s$. Then $\langle I \rangle (t) = \int_0^t g^2(s)ds$; this is the quadratic variation of an Itô integral for a simple integrand. This result, which was proven in the lecture, also holds even when g is not an elementary process.

8. The Itô's formula was presented and motivated by the Taylor's formula truncated after the second term. The Itô's differentiation rule for $f(W_t)$

says the following: Suppose f is twice differentiable and W_t is a BM then

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt$$

or in integral form

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds.$$

9. It was noted that we have "intuitive meaning" for $df(W_t)$, dW_t and dt in the previous statement but no solid definition. The equation for $df(W_t)$ in item #8 becomes "mathematically respectable" only after we integrate it since we have definitions for both integrals in its right hand side. However, we use the differential form for calculation purposes.

10. For the geometric Brownian motion (GBM) with dynamics given by the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$, we applied Itô's lemma by considering the function $f(t, S_t)$. We showed in class that the solution to this SDE is $S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$.