FM9561B – 10 – 14 February 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. We analysed the Vasiček model for the intererest rate process specified by the SDE: $dr_t = a(b - r_t)dt + \sigma dW_t$ under a risk-neutral measure. The parameters a, b and σ refer to the speed of mean-reversion, mean-reverting level and volatility, respectively.

2. By Itô's lemma $r_t = \mu_t + \sigma \int_0^t e^{a(s-t)} dW_s$ where μ_t is a deterministic function and $\mu_t := E[r_t] = e^{-at}\left[r_0 + b(e^{at} - 1)\right]$. In general, if $f(v)$ is deterministic (i.e., a function only of t), $\int_0^t f(v)dW_v$ is Gaussian (see Karatzas and Shreve, for example, for a proof).

3. We note that r_t is a Gaussian random variable with variance $\sigma_t^2 = \sigma^2 \left(\frac{1 - e^{-2at}}{2a}\right)$. Since normal RVs can become negative with positive probability, this is considered to be the weakness of the Vasiček model. However, its simplicity and tractability validate its discussion.

4. As an exercise, one may prove that
$E\left[-\int_t^T r_u du\right] = -\frac{r_t - b}{a}\left(1 - e^{-a(T-t)}\right) - b(T-t)$
and $\text{Var}\left[-\int_t^T r_u du\right] = \frac{\sigma^2}{2a^3}\left(2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}\right)$.

5. Consequently the price $B(t,T)$ of a default-free zero-coupon bond is given by

$$\begin{aligned} B(t,T) &= E\left[\exp\left(-\int_t^T r_u du\right)\bigg| \mathcal{F}_t\right] = E\left[\exp\left(-\int_t^T r_u du\right)\bigg| r_t\right] \\ &= \exp\left(E\left[-\int_t^T r_u(r_t)du\right] + \frac{1}{2}\text{Var}\left[-\int_t^T r_u(r_t)du\right]\right). \end{aligned}$$

Note that we make use of the fact that r_t is Markov.

6. Using the expressions for the mean and variance of $-\int_t^T r_u du$ in #4, we have the functional form for the bond price given by $B(t,T) = \exp(-A(t,T)r_t + D(t,T))$ where $A(t,T) = \dfrac{1-e^{-a(T-t)}}{a}$ and $D(t,T) = \left(b - \dfrac{\sigma^2}{2a^2}\right)[A(t,T) - (T-t)] - \sigma^2 \dfrac{A(t,T)^2}{4a}$.

7. Since the yield $y(t,T)$ can be represented as $y(t,T) := -\dfrac{\log B(t,T)}{T-t} = \dfrac{1}{T-t}A(t,T)r_t - \dfrac{1}{T-t}D(t,T)$, we say that $y(t,T)$ is affine in r_t and the Vasiček model is an example of an affine term structure model or an exponential affine bond price model.

8. We discussed an alternative approach in the Vasicek's bond valuation via the bond price PDE. Here, the solution will not depend on the distributional property of r_t, which was the basis in the direct approach of calculating the conditional expectation.

9. Using the fact that under the Vasiček model r_t is Markov and $B(t,T,r) := E\left[\exp\left(-\int_t^T r_u(r)du\right)\bigg| r_t = r\right]$ with

$r_u = e^{-a(u-t)}\left[r + b(e^{a(u-t)} - 1) + \sigma \int_t^u e^{a(s-t)} dW_s\right]$, we show that $\frac{\partial B}{\partial r} = -A(t,T)B(t,T,r)$. Thus,

$$B(t,T,r) = C(t,T)\exp(-A(t,T)r), \tag{1}$$

where $A(t,T) = \frac{1 - e^{-a(T-t)}}{a}$. Our aim is to find the deterministic function $C(t,T)$ (and independent of r) in equation (1). This can be obtained from the bond price PDE, which will be established in the next lecture.