FM9561B – 17 – 21 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. We have two approaches in term structure modelling: short rate approach and HJM or forward rate approach. Under the short rate approach, we start with an SDE for a short rate process r_t and then obtain the bond price. From the bond price, we can obtain both the yield rate and forward rate.

   On the other hand, under the HJM methodology, we start with the dynamics of the forward rate process. Then we obtain the bond price from which we can derive the yield rate.

   To recover $r(t)$ from $f(t,T)$, we evaluate $f(t,T)$ at $T = t$.

   The above summary of 2 modelling approaches was given in the lecture. Previously, a term-structure data compiled by the Bank of Canada plotted as a surface (being a function of time, maturity and yield) was highlighted as the financial variable, whose evolution and dynamics are what the term structure theory aims to accurately and efficiently capture.

2. A list of common one-factor and two-factor short rate models was given in the lecture. Majority of the models proposed in the literature are special cases of the parametric form of the generalised model presented in the class.

3. The characterisation of the exponential affine models was presented in the lecture:

Duffie and Kan (1996): The zero-coupon bond price is exponential affine iff the short rate dynamics, i.e., drift and variance rate, must be of linear/Gaussian or square root/affine form.
Elliott and van der Hoek (2001): If the short rate dynamics are of linear Gaussian or square root affine form, then the bond price is exponential affine. *This statement was proved using the forward measure approach, which is a different technique from the one used in Duffie and Kan (1996).*

4. When one cannot solve an SDE explicitly, it is possible to simulate its trajectories through an Euler discretisation scheme. Consider the SDE $dX_t(\omega) = f(X_t(\omega))dt + \sigma(X_t(\omega))dW_t$. Integrate this equation between s and $s + \Delta s$:

$$X_{s+\Delta s}(\omega) = X_s(\omega) + \int_s^{s+\Delta s} f(X_t(\omega))dt + \int_s^{s+\Delta s} \sigma(X_t(\omega))dW_t(\omega).$$

The Euler scheme consists of approximating this integral equation by

$$\bar{X}_{s+\Delta s}(\omega) = \bar{X}_s(\omega) + f(\bar{X}_s(\omega))\Delta s + \sigma(\bar{X}_s(\omega))(W_{s+\Delta}(\omega) - W_s(\omega))$$

with $\bar{X}_0(\omega) = x_0$. If we apply this formula iteratively for a given set of s's say $s = s_1,\ s_2, \ldots, s_m,\ s_1 = 0$ and $s_m = T$, we obtain a discretised approximation $\bar{X}$ of the solution of X for the above SDE.

5. A more refined scheme is called Milstein scheme. This will not be reviewed here. We, however, hint at the fact that when the diffusion coefficient is deterministic, i.e., $\sigma(X_t, \omega)) = \sigma(t)$, a deterministic function of time, the Euler and Milstein schemes coincide. When possible, apply the Euler scheme to SDEs with deterministic diffusion coefficients since this ensures the same convergence with that of the Milstein scheme.

6. The Euler discretisation scheme can be useful for Monte Carlo simulation. Suppose we need to compute the expected value of a function of

the solution X of the SDE in #4, say for simplicity,

$$E^Q\left[\phi(X_{s_1}(\omega),\ldots,X_{s_m}(\omega))\right],\quad s_1=0,\ s_m=T.$$

The evaluation of the above SDE is typical in pricing path-dependent pay-offs in quantitative finance.

Assume that the times s are close to each other. We compute an approximation of this expectation as follows:

(i) Select the number N of scenarios for the Monte Carlo method.
(ii) Set the initial value to $\bar{X}_0^j = x_0$ for all scenarios, $j=1,\ldots,N$.
(iii) Set $k=1$.
(iv) Set $s=s_k$ and $\Delta s = s_{k+1}-s_k$ so that $s+\Delta s = s_{k+1}$.
(v) Generate N new realisations $\Delta W^j,\ j=1,\ldots,N$ of a standard Gaussian distribution N(0,1) multiplied by $\sqrt{\Delta s}$, thus simulating the distribution of $W_{s+\Delta s}(\omega)-W_s(\omega)$.
(vi) Apply the approximation formula in #4 for each scenario $j=1,\ldots,N$ with the generated shocks:

$$\bar{X}^j_{s+\Delta s} = \bar{X}^j_s + f(\bar{X}^j_s)\Delta s + \sigma(\bar{X}^j_s)\Delta W^j.$$

(vii) Store $\bar{X}^j_{s+\Delta s}$ for all j.
(viii) If $s+\Delta s = s_m$ then stop, otherwise increase k by one and start again from point (iv).
(ix) Approximate the expected value by $\dfrac{\sum_{j=1}^N \phi(X_{s_1}(\omega),\ldots,X_{s_m}(\omega))}{N}$.

7. For the Vasiček model $dr_t = a(b-r_t)dt + \sigma dW_t$, one can simulate the paths of the interest rate process in two ways:
Doing it naively by using

$$r_{t_{i+1}} = r_{t_i} - a(b-r_{t_i})\Delta t_{i+1} + \sigma\sqrt{\Delta t_{i+1}}\epsilon_{i+1}$$

or doing it properly using

$$r_{t_{i+1}} = e^{-a(t_{i+1}-t_i)}r_{t_i} + b\left(1-e^{-a(t_{i+1}-t_i)}\right) + \sigma\sqrt{\frac{1}{2a}\left(1-e^{-2a(t_{i+1}-t_i)}\right)}\epsilon_{i+1},$$

where the second discretisation is based on the analytic solution of r_t and $\epsilon_{i+1} \sim N(0,1)$.

Remark: We showed that the Euler scheme for the "naive" discretisation assumes the approximation $e^x \approx 1 + x$.

Similarly, for the geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dW_t$, one has the choice to employ the discretisation $S_{k+1} - S_k = \mu S_k \Delta t_k + \sigma\sqrt{\Delta t_{k+1}}\epsilon_{k+1}$ or $S_{k+1} = S_k \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\Delta t_k + \sigma\sqrt{\Delta t_k}\epsilon_{t_{k+1}}\right]$. The second equation will be a better choice since it is based on the closed-form solution of S_t.

8. The Cholesky's decomposition, which was discussed in the lecture, must be applied when correlated samples are needed in the simulation of Brownian motion sample paths.