FM9561B – 20 – 24 January 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. For the GBM $S_t = S_0 + \int_0^t \mu_u S_u du + \int_0^t \sigma_u S_u dW_u$, its quadratic variation is given by $\langle S \rangle_t = \int_0^t \sigma_u^2 S_u^2 du$.

2. We considered a model for the wealth process of an investor who has investment in a risky asset evolving as GBM and in a money market (or bank account). We set up (and determine the dynamics) of the replicating portfolio that will synthesise the price of a European contingent claim.

3. We examined an Itō process X_t which has the general form:

$$X_t = X_0 + \int_0^t H_s ds + \int_0^t K_s dW_s$$

   where H_s and K_s are stochastic processes such that $E\left[\int_0^t H_s^2 ds\right] < \infty$ and $E\left[\int_0^t K_s^2 ds\right] < \infty$.

4. The generalised Itō's lemma in integral form was formulated for the one-dimensional case.

Using the dynamics of X_t in no. 2, Itō's lemma states that if $f : [0,\infty] \times I\!R \to I\!R$ is $C^{1,2}$ then $f(t, X_t)$ is an Itô process with dynamics

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \frac{\partial}{\partial s} f(s,X_s)ds + \int_0^t \frac{\partial}{\partial x} f(s,X_s)dX_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(s,X_s)d\left\langle X\right\rangle_s \\ &= f(0,X_0) + \int_0^t \frac{\partial}{\partial s} f(s,X_s)ds + \int_0^t \frac{\partial}{\partial x} f(s,X_s)(H_s ds + K_s dW_s) \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(s,X_s)K_s^2 ds. \end{aligned}$$

5. In an attempt to price an option, we constructed a replicating portfolio that duplicates the price of an option. Suppose at time 0, the price of the risky asset is S_0. The investor can start at time 0 with initial wealth $Z_0 := V(0, S_0)$. We showed that the investor should **hedge** by buying $\Delta(t) = V_S(t, S)$ units of S_t. At time T his wealth will be $Z_T = V(T, S_T) = h(S_T)$, where in general $h(\cdot)$ is the pay-off of the contingent claim at time T. Therefore, the value of the option at time t is $Z_t = V(t, S_t)$.

6. It was shown in class that if V denotes the price of a contingent claim $h(S_T)$ in the Black-Scholes framework then it satisfies the PDE $V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV$ with terminal condition $V(T, S_T) = h(S_T)$. Prices of financial instruments can be calculated by solving this PDE with respect to a boundary condition that describes the pay-off of the instrument at time T.

   Remark: *It was pointed out that the above PDE has a probabilistic solution in terms of a conditional expectation. The link between PDEs and conditional expectations is contained in the Feynman-Kac's theorem.*

7. The Feynman-Kac's theorem provides a probabilistic solution to a certain class of PDEs. Suppose the PDE $V_t + \mu(t,x)V_x + \frac{1}{2}\sigma(t,x)^2 V_{xx} - r(t,x)V = 0$ has a boundary condition $H(T,x)$. Then the solution of this PDE is $E\left[\exp\left(-\int_t^T r(u,x)du\right)H(T,x)\middle|\mathcal{F}_t\right]$ where the expectation is taken with respect to the measure which defines the process X, i.e., $dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t$.

   Note that this conditional expectation also gives the risk-neutral valuation formula for a contingent claim H.

8. In the lecture, we also discussed the problem of solving the mean and variance of a stochastic process without having to solve explicitly the SDE satisfied by this given stochastic process. The technique was illustrated using the Cox-Ingersoll-Ross interest rate model. In this technique we liberally use the martingale property of an Itô integral. A linear ODE arises in the solution of this problem.

9. ($d-$dimensional BM:) A $d-$dimensional BM is a process $\mathbf{W}_t = (W_t^1, W_t^2, \ldots, W_t^d)$ with the following properties:
   (i) Each W_t^j is a one-dimensional BM.
   (ii) If $i \neq j$ then the process W_t^i and W_t^j are independent.