FM9561B – 24 – 28 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. The maximum likelihood estimation technique to calculate model parameters using a dataset was illustrated under the Vasiček model. Consider the model $dr_t = \alpha(\mu - r_t)dt + \sigma dW_t$ and suppose its discretised version is $r_{t+\Delta t} = a + br_t + \sigma\sqrt{\Delta}\epsilon_t$ where $a = \alpha\mu\Delta t$ and $b = 1 - \alpha\Delta t$. Thus, $r_{t_{i+1}} \sim N(a + br_t, \sigma^2\Delta t)$. So, one can write the log-likelihood $L(\theta)$ for $\theta = (a,\ b,\ \sigma)$. This is given by

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{N-1} (2\pi\sigma^2\Delta t_i)^{-\frac{1}{2}} \exp\left(\frac{r_{t_{i+1}} - (a + br_{t_i})}{\sigma\sqrt{\Delta t_i}}\right) \\ &= (2\pi\sigma^2\Delta t)^{-(N-1)/2} \exp\left[-\frac{1}{2}\sum_{i=1}^{N-1}\left(\frac{r_{t_{i+1}} - (a + br_{t_i})}{\sigma\sqrt{\Delta t}}\right)^2\right], \end{aligned}$$

   where the second equation is a simplication when $\Delta t_i = \Delta t$, a constant. Setting the partial derivatives with respect to a, b and σ to 0 would yield a system of equations that need to be solved to obtain the estimates $\widehat{a}$, $\widehat{b}$ and $\widehat{\sigma}$.

2. **Review of the HJM Model under a risk-neutral measure**
   • Dynamics of the forward rate $f(t,T) : df(t,T) = \sigma(t,T)\sigma^*(t,T)dt + \sigma(t,T)dW_t$ where $\sigma^*(t,T) = \displaystyle\int_t^T \sigma(t,u)du$.

   • The short rate is given by $r_t = f(t,t)$.

   • The bond price (in terms of $f(t,T)$) is given by $B(t,T) = \exp\left[-\displaystyle\int_t^T f(t,u)du\right]$.

• The bond price dynamics is given by $dB(t,T) = r_t B(t,T)dt - \sigma^*(t,T)B(t,T)dW_t$.

• To implement the HJM, one needs to specify $\sigma(t,T), \quad 0 \le t \le T$. A simple choice one may consider is $\sigma(t,T) = \sigma f(t,T)$, where $\sigma > 0$, σ being the constant volatility of the forward rate. We demonstrated in class that this choice is not possible since the solution $f(t,T)$ explodes at some time t.

3. **The Brace-Gatarek-Musiela (BGM) Model**

• The HJM model is re-parametrised to obtain the BGM model. Apparently, BGM is a subclass of HJM model.

• New variables:
Current time is t and time to maturity is $\tau := T - t$.
Forward rates: $r(t,\tau) = f(t,t+\tau)$ so that $r(t,0) = f(t,t) = r_t$ and $\frac{\partial}{\partial \tau} r(t,\tau) = \frac{\partial}{\partial T} f(t,t+\tau)$.
Bond prices: $D(t,\tau) = B(t,t+\tau) = \exp\left(-\int_0^\tau f(t,t+u)\right) = \exp\left(-\int_0^\tau r(t,u)du\right)$.
$\dfrac{\partial}{\partial \tau} D(t,\tau) = \dfrac{\partial}{\partial T} B(t,t+\tau) = -r(t,\tau)D(t,\tau)$.

4. LIBOR: Fix $\delta > 0$. \$ $D(t,\delta)$ invested at time t in a $(t+\delta)-$maturity bond that grows to \$1 at time $t+\delta$. $L(t,0)$ is defined to be the corresponding rate of simple interest. We showed in the lecture that

$$L(t,0) = \frac{\exp\left[\int_0^\delta r(t,u)du\right] - 1}{\delta}.$$

5. FORWARD LIBOR: Let $\delta > 0$ be fixed. At time t, agree to invest \$ $\dfrac{D(t,t+\delta)}{D(t,\delta)}$ at time $t+\tau$, with payback of \$1 at time $t+\tau+\delta$. The

forward LIBOR $L(t,\tau)$ is defined to be the simple (forward) interest rate for this investment. We showed in class that

$$L(t,\tau) = \frac{\exp\left(\int_{\tau}^{\tau+\delta} r(t,u)du\right) - 1}{\delta}.$$

Consequently,

$$f(t,t+\tau) = r(t,\tau) = \lim_{\delta\to 0^+} \frac{\exp\left(\int_{\tau}^{\tau+\delta} r(t,u)du\right) - 1}{\delta}.$$

6. $r(t,\tau)$ is the continuously compounded rate. $L(t,\tau)$ is the simple rate over a period of duration δ. We cannot have a log-normal model for $r(t,\tau)$ since solutions explode (as previously shown). For a fixed positive δ, we can have a log-normal model for $L(t,\tau)$.

7. The long rate is determined by long maturity bond prices. Let n be a large fixed positive integer so that $n\delta$ could be 20 to 30 years. Then, $\frac{1}{D(t,n\delta)} = \exp\left(\int_0^{n\delta} r(t,u)du\right)$. In terms of the forward LIBOR, the long rate is

$$\frac{1}{n\delta}\log\frac{1}{D(t,n\delta)} = \frac{1}{n\delta}\sum_{k=1}^{n}\log[1+\delta L(t,(k-1)\delta)].$$

8. We consider the pricing of a swap. Suppose $T_0 \geq 0$ and $T_1 = T_0 + \delta$, $T_2 = T_0 + 2\delta$, ..., $T_n = T_0 + n\delta$. The swap is the series of payments $\delta(L(T_k,0) - c)$ at time T_{k+1}, $k = 0, 1, \ldots, n-1$. For $0 \leq t \leq T_0$, the value of the swap is

$$\sum_{k=0}^{n-1} E\left[\frac{\beta(t)}{\beta(T_{k+1})}\delta(L(T_k,0)-c)\middle| \mathcal{F}_t\right].$$

The value of c, which would make the value of the swap contract at time 0 is called the *swap rate*. The calculation of this quantity was also shown in the lecture.

9. Another important topic in stochastic modelling of interest rates is the valuation of interest rate options such as caps and floors, which was also discussed in the lecture. When the LIBOR rate is assumed to follow lognormal dynamics, analytic solutions to caps, floors and collars are available, and such framework is called the Black's model.