Statistical Sciences 3520B

SOLUTIONS TO THE UNFINISHED EXAMPLE
07 March 2014 Lecture

Consider a 2-step trinomial non-recombining lattice tree model. For each step, there are three possibilities for the stock price: an up movement (u), a down movement (d) or no movement (n).

(a) Write down the set or sample space, Ω, containing all possible outcomes for this 2-step trinomial tree. If we consider the collection of all subsets of Ω, how many subsets are there in this collection?

(b) Suppose the respective probabilities of events $\{u\}$ and $\{d\}$ are 3/7 and 2/7. Define or construct the probability measure for each individual element $\omega \in \Omega$.

(c) Write down the σ-algebra or σ-field $\{\Im_i\}$ keeping track the outcomes for each time step i = 0, 1, 2.

(d) Consider the Ω given in (a). Under the trinomial asset pricing model suppose $S_0 = 50$, $d = 10/11$ and $u = 12/11$; clearly, $\{S_i\}$ is a stochastic process, i.e., S_i's are random variables for i = 0, 1, 2. Find $S_1(\varpi)$, i.e., what is the function $S_1(\varpi)$?

(e) Consider the interval $B=\left[2\pi^{e}-1,\ 2e^{\pi}+1\right]$. What is $S_1^{-1}(B)$? Recall that S_1 is a random variable and by definition it maps Ω into $\Re$.

Answer:

(a) All possible outcomes of tossing a coin three times can be described by the set $\Omega = \{uu, un, ud, nu, nn, nd, du, dn, dd\}$. Since there are 9 elements in Ω, there would be 2^9 = 512 subsets of Ω contained in the collection.

(b) For the individual elements of Ω, we have

$P(uu) = (3/7)^2 = 9/49$ $\quad$ $P(nn) = (2/7)^2 = 4/49$

$P(un) = (3/7)(2/7) = 6/49$ $\quad$ $P(nd) = (2/7)^2 = 4/49$

$P(ud) = (3/7)(2/7) = 6/49$ $\quad$ $P(du) = (2/7)(3/7) = 6/49$

$P(nu) = (2/7)(3/7) = 6/69$ $\quad$ $P(dn) = (2/7)^2 = 4/49$

$$P(dd) = (2/7)^2 = 4/49$$

Note that $\sum_{\omega\in\Omega} P\{\omega\} = 1$.

(c) $\Im_0 = \{\phi, \Omega\}$ signifying we do not know anything yet and we note that $\phi^c = \Omega \in \Im_0$ since $\Im$ is closed under set complementation.

$$\Im_1 = \left\{ \begin{array}{l} \phi, \Omega, \{uu, un, ud\}, \{nu, nn, nd\}, \{du, dn, dd\}, \text{and all sets which} \\ \text{can be built by taking unions and set complementation (i.e., intersections) of these} \end{array} \right\}$$

which signifies that we have either *u, n* or *d* on the first-time step.

$\Im_2 = \Im =$ The collection of all subsets of Ω.

(d) For S_1, we have

$S_1(uu) = S_1(un) = S_1(ud) = 50(12/11) = 54.55$
$S_1(nu) = S_1(nn) = S_1(nd) = 50(1)(1) = 50.00$
$S_1(du) = S_1(dn) = S_1(dd) = 50(10/11) = 45.45$

Therefore, $$S_2(\omega) = \begin{cases} 54.55 & \text{if} \quad \omega \text{ is uu or un or ud} \\ 50 & \text{if} \quad \omega \text{ is nu or nn or nd} \\ 45.45 & \text{if} \quad \omega \text{ is du or dn or dd} \end{cases}$$

(e) We consider the interval $\left[2\pi^e - 1,\ 2e^\pi + 1\right]$ = [43.9183, 47.2814]. The pre-image under S_1 of the interval [43.9183., 47.2814] is defined to be $\{\omega \in \Omega : S_1(\omega) \in [43.9183, 47.2814]\} = \{\omega \in \Omega : 43.9183 \le S_1 \le 47.2814\} = \{du, dn, dd\}$.