Statistical Sciences 3520B

SOLUTIONS TO PRACTICE PROBLEMS
Assignment No. 2

You will learn and benefit more if you attempt solving these problems first before looking at their solutions.

BINOMIAL OPTION PRICING PROBLEMS

Problem 12.2

Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

Answer: In the no-arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on the portfolio equal to the risk-free interest rate, we are able to value the option. When we use risk-neutral valuation, we first choose probabilities for the branches of the tree so that the expected return on the stock equals the risk-free interest rate. We then value the option by calculating its expected payoff and discounting this expected payoff at the risk-free interest rate.

Problem 12.3

What is meant by the delta of a stock option?

Answer: The delta of a stock option measures the sensitivity of the option price to the price of the stock when small changes are considered. Specifically, it is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

Problem 12.5

A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100?

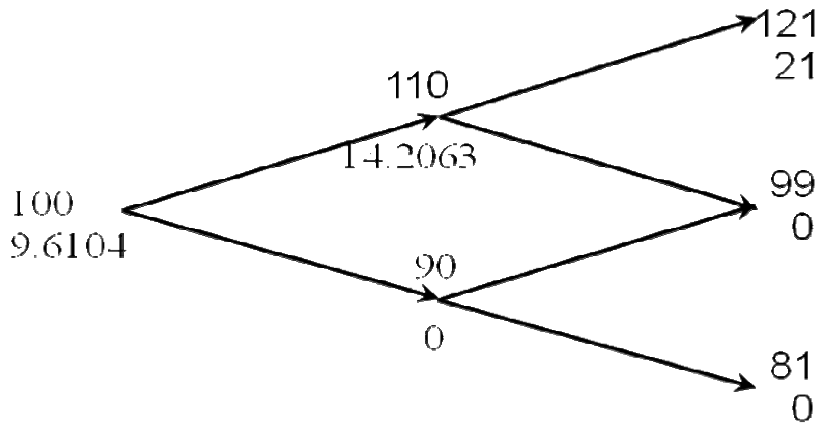
Answer: In this case $u = 1.10$, $d = 0.90$, $\Delta t = 0.5$, and $r = 0.08$, so that

$$p = \frac{e^{0.08\times0.5} - 0.90}{1.10 - 0.90} = 0.7041$$

The tree for stock price movements is shown below. We can work back from the end of the tree to the beginning, as indicated in the diagram, to give the value of the option as \$9.61. The option value can also be calculated directly using the risk-neutral valuation formula:

$$[0.7041^2 \times 21 + 2 \times 0.7041 \times 0.2959 \times 0 + 0.2959^2 \times 0]e^{-2\times 0.08\times 0.5} = 9.61$$

or \$9.61.



Binomial tree for problem 12.5

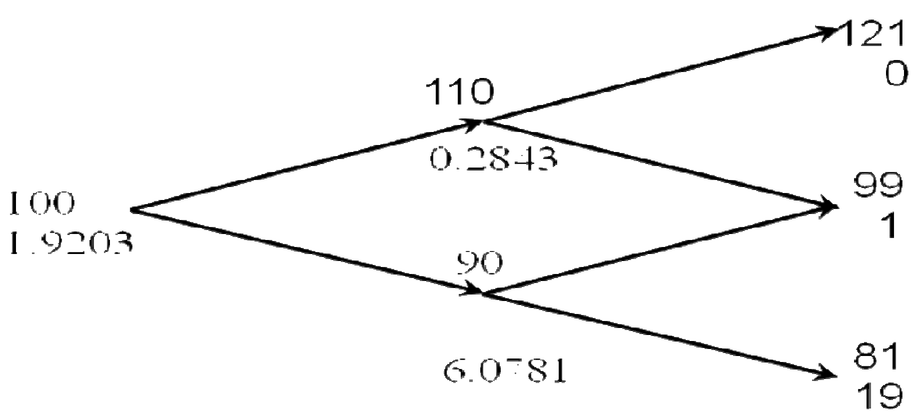
Problem 12.6

For the situation considered in Problem 12.5, what is the value of a one-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put–call parity.

Answer: The figure below shows how we can value the put option using the same tree as in Problem 12.5. The value of the option is \$1.92. The option value can also be calculated directly from the risk-neutral valuation formula:

$$e^{-2\times 0.08\times 0.5}[0.7041^2 \times 0 + 2 \times 0.7041 \times 0.2959 \times 1 + 0.2959^2 \times 19] = 1.92$$

or \$1.92. The stock price plus the put price is $100 + 1.92 = \$101.92$. The present value of the strike price plus the call price is $100e^{-0.08\times 1} + 9.61 = \$101.92$. These are the same, verifying that put–call parity holds.



Binomial tree for problem 12.6

Problem 12.8

Consider the situation in which stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

Answer: The riskless portfolio consists of a short position in the option and a long position in Δ shares. Because Δ changes during the life of the option, this riskless portfolio must also change.

Problem 12.10

A stock price is currently $80. It is known that at the end of four months it will be either $75 or $85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a four-month European put option with a strike price of $80? Use no-arbitrage arguments.

Answer: At the end of four months the value of the option will be either $5 (if the stock price is $75) or $0 (if the stock price is $85). Consider a portfolio consisting of:

$$-\Delta \quad : \quad \text{shares}$$

$$+1 \quad : \quad \text{option}$$

(Note: The delta, Δ of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either -85Δ or $-75\Delta+5$ in four months. If

$$-85\Delta = -75\Delta + 5$$

i.e.,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 42.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.5 \times 80 + f$$

where f is the value of the option. Since the portfolio is riskless

$$(0.5 \times 80 + f)e^{0.05 \times 4/12} = 42.5$$

i.e.,

$$f = 1.80$$

The value of the option is therefore $1.80.

This can also be calculated directly using the risk-neutral valuation principle. With $u = 1.0625$ and $d = 0.9375$, the risk-neutral probability p (or q in our notation in class), we have

$$p = \frac{e^{0.05 \times 4/12} - 0.9375}{1.0625 - 0.9375} = 0.6345$$

$1 - p = 0.3655$ and

$$f = e^{-0.05 \times 4/12} \times 0.3655 \times 5 = 1.80$$

Problem 12.11

A stock price is currently \$40. It is known that at the end of three months it will be either \$45 or \$35. The risk-free rate of interest with quarterly compounding is 8% per annum. Calculate the value of a three-month European put option on the stock with an exercise price of \$40. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

Answer: At the end of three months the value of the option is either \$5 (if the stock price is \$35) or \$0 (if the stock price is \$45).

Consider a portfolio consisting of:

$$-\Delta \quad : \quad \text{shares}$$

$$+1 \quad : \quad \text{option}$$

(Note: The delta, Δ, of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either $-35\Delta+5$ or -45Δ. If:

$$-35\Delta+5=-45\Delta$$

i.e.,

$$\Delta=-0.5$$

the value of the portfolio is certain to be 22.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is

$$-40\Delta+f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(40\times 0.5+f)\times 1.02=22.5$$

Hence

$$f=2.06$$

i.e., the value of the option is \$2.06.

This can also be calculated using risk-neutral valuation. Suppose that p (q in our notation in the class) is the probability of an upward stock price movement in a risk-neutral world. We must have

$$45p+35(1-p)=40\times 1.02$$

i.e.,

$$10p=5.8$$

or:

$$p=0.58$$

The expected value of the option in a risk-neutral world is:

$$0\times 0.58+5\times 0.42=2.10$$

This has a present value of

$$\frac{2.10}{1.02}=2.06$$

This is consistent with the no-arbitrage answer.

Problem 12.14

A stock price is currently \$25. It is known that at the end of two months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of two months. What is the value of a derivative that pays off S_T^2 at this time?

Answer: At the end of two months the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

$$+\Delta \;:\; \text{shares}$$

$$-1 \;:\; \text{derivative}$$

The value of the portfolio is either $27\Delta - 729$ or $23\Delta - 529$ in two months. If

$$27\Delta - 729 = 23\Delta - 529$$

i.e.,

$$\Delta = 50$$

the value of the portfolio is certain to be 621. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$50 \times 25 - f$$

where f is the value of the derivative. Since the portfolio must earn the risk-free rate of interest

$$(50 \times 25 - f)e^{0.10 \times 2/12} = 621$$

i.e.,

$$f = 639.3$$

The value of the option is therefore \$639.3.

This can also be calculated directly from the risk-neutral valuation principle. With $u = 1.08$, $d = 0.92$, the risk-neutral probability p (q in our notation in the class),

$$p = \frac{e^{0.10 \times 2/12} - 0.92}{1.08 - 0.92} = 0.6050$$

and

$$f = e^{-0.10 \times 2/12}(0.6050 \times 729 + 0.3950 \times 529) = 639.3$$

Problem 12.15

Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is one month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

Answer: In this case

$$a = e^{(0.05 - 0.08) \times 1/12} = 0.9975$$

$$u = e^{0.12\sqrt{1/12}} = 1.0352$$

$$d = 1/u = 0.9660$$

$$p = \frac{0.9975 - 0.9660}{1.0352 - 0.9660} = 0.4553$$ (The p here is q in our notation in the class).

♣REQUIRED PROBLEM #1 [4 points]
Additional Problem 1

A stock price is currently \$25. It is known that at the end of 4 months it will be either \$30 or \$21. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a 4-month European call option with an exercise price of \$24. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answer. [4 points]

Solution given in a separate sheet.

♣REQUIRED PROBLEM #2 [5 points]
Additional Problem 2

In a two-period binomial model with r =1% per period, the current stock price is \$100, and u =1.02 and d =0.98. Consider an option that expires after two periods, and pays the value of the squared stock price, $S(t)^2$, if the stock price $S(t)$ is higher than \$100 when the option is exercised. Otherwise (when $S(t)$ is less than or equal to \$100), the option pays zero. If the option under consideration is an American-type, find its price. [5 points]

Solution given in a separate sheet.

VALUE-AT-RISK PROBLEMS

Problem 21.1

Consider a position consisting of a \$100,000 investment in asset A and a \$100,000 investment in asset B. Assume that the daily volatilities of both assets are 1% and that the coefficient of correlation between their returns is 0.3. What is the 5-day 99% VaR for the portfolio?

Answer: The standard deviation of the daily change in the investment in each asset is \$1,000. The variance of the portfolio's daily change is

$$1,000^2 + 1,000^2 + 2\times 0.3\times 1,000\times 1,000 = 2,600,000$$

The standard deviation of the portfolio's daily change is the square root of this or \$1,612.45. The standard deviation of the 5-day change is

$$1,612.45\times\sqrt{5} = \$3,605.55$$

From the tables of $N(x)$ we see that $N(-2.33) = 0.01$. This means that 1% of a normal distribution lies more than 2.33 standard deviations below the mean. The 5-day 99 percent value at risk is therefore $2.33 \times 3,605.55 = \$8,401$.

Problem 21.6

Suppose a company has a portfolio consisting of positions in stocks, bonds, foreign exchange, and commodities. Assume there are no derivatives. Explain the assumptions underlying (a) the linear model and (b) the historical simulation model for calculating VaR.

Answer: The linear model assumes that the percentage daily change in each market variable has a normal probability distribution. The historical simulation model assumes that the probability distribution observed for the percentage daily changes in the market variables in the past is the probability distribution that will apply over the next day.

♣REQUIRED PROBLEM #3 [4 points]

Additional Problem 3

Brian Griffin is a small investor who has 70% of his portfolio invested in a market-index fund, and 30% in a small-stocks fund. The mean monthly return rate of the market-index fund is 1.5% with a standard deviation of 0.9%. The small-stocks fund has the mean monthly return rate of 2.2% with standard deviation of 1.2%. The correlation between the two funds is 0.13. Assume normal distribution for the return rates. What is the monthly VaR at 99% level for Brian's portfolio if the portfolio value today is $100,000?

Solution given in a separate sheet.

BASIC ELEMENTS OF STOCHASTIC PROCESSES

♣REQUIRED PROBLEM #4 [7 points]

Additional Problem 4

Consider the experiment of drawing (with replacement) a ball three times from a bag containing a blue ball (B) and a red ball (R).

(a) Write down the set or sample space, Ω, containing all possible outcomes. If we consider the collection of all subsets of Ω, how many sets are there in this collection?

(b) Suppose the probability of getting B is 3/7 and the probability of getting R is 4/7. Define or construct the probability measure for each individual element $\omega \in \Omega$.

(c) Write down the σ-algebra or σ-field $\{\Im_i\}$ keeping track the outcomes of each drawing for i = 0, 1, 2, 3.

(d) Let Ω be given as in (a) and consider the binomial asset pricing model where S_0 = 20, d =4/5 and u = 5/4 so that $\{S_i\}$ is a stochastic process, i.e., S_i's are random variables for i = 0, 1, 2, 3. Find $S_2(\varpi)$, i.e., what is the function $S_2(\varpi)$?

(e) Consider the interval $\left[e^{\pi}\text{-}9,\ \pi^{e}+9\right]$. What is the pre-image under S_2 of this interval? Recall that S_2 is a random variable and by definition it maps Ω into $\Re$.
Solution given in a separate sheet.