Statistical Sciences 3520B
Winter 2014
SOLUTIONS TO REQUIRED PROBLEMS
Assignment No. 2

♣REQUIRED PROBLEM #1 [4 points]
Additional Problem 1

A stock price is currently \$25. It is known that at the end of 4 months it will be either \$30 or \$21. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a 4-month European call option with an exercise price of \$24. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answer.

Answer:

At the end of four months the value of the option will be either \$6 (if the stock price is \$30) or \$0 (if the stock price is \$21). Consider a portfolio consisting of:

$$\left.\begin{array}{ll} \Delta: & \text{shares} \\ -1: & \text{option} \end{array}\right\} \quad 0.5$$

The value of the portfolio is either 30Δ-6 or 21Δ in four months. If

$$30\Delta - 6 = 21\Delta \quad 0.5$$

i.e.,

$$\Delta = 2/3 \quad 0.5$$

the value of the portfolio is certain to be 14. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$25\Delta - f = 25(2/3) - f$$

where f is the value of the option. Since the portfolio is riskless 0.5

$$[25(2/3) - f]e^{(0.12)(4/12)} = 14$$

i.e.,

$$\boxed{f = \$3.2156.} \quad 0.5$$

The value of the option is therefore \$3.22.

0.5

This can also be calculated using the risk-neutral valuation approach, where u= 1.20 and d = 0.84 so that

$$q = \frac{e^{0.12(4/12)} - 0.84}{1.20 - 0.84} = 0.557807706 \quad 0.5$$

and 1- q = 0.442192294. Therefore,

$$\boxed{f = e^{-0.12(4/12)}(0.557807706)(6) = \$3.2156.}$$

0.5

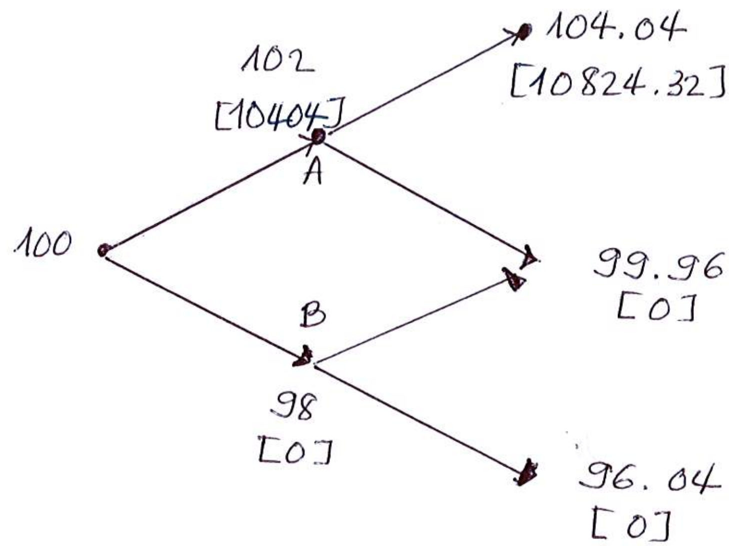
♣REQUIRED PROBLEM #2 [5 points]

Additional Problem 2

In a two-period binomial model with $r = 1\%$ per period, the current stock price is \$100, and $u = 1.02$ and $d = 0.98$. Consider an option that expires after two periods, and pays the value of the squared stock price, $S(t)^2$, if the stock price $S(t)$ is higher than \$100 when the option is exercised. Otherwise (when $S(t)$ is less than or equal to \$100), the option pays zero. If the option under consideration is an American-type, find its price.

Answer:

We present the corresponding tree for the stock price.



2

In the brackets we record the payoff of the American option if we exercise it in the corresponding node. In the final nodes, the payoff is equal to the payoff of a European-style option.

In node B, the payoff and the value of the unexercised option is 0. 1

In node A, if we exercise the option, the payoff is $102^2 = 10{,}404$. 0.5 We have to compare this payoff with the value of the unexercised option. For that, we need t compute the risk-neutral probability q as

$$q = \frac{(1+r)-d}{u-d} = \frac{1.01-0.98}{1.02-0.98} = 0.75 .$$

0.5

The value of the unexercised option at that node is

$$\frac{1}{1.01}(0.75)(10{,}824.32) = 8{,}037.86 .$$

Therefore, it is optimal to exercise early at this node (node A). 0.5

We compute the price of the option at the initial moment as

$$\frac{1}{1.01}(0.75)(10{,}404) = \$7{,}725.743 \quad 0.5$$

♣REQUIRED PROBLEM #3 [4 points]

Additional Problem 3

Brian Griffin is a small investor who has 70% of his portfolio invested in a market-index fund, and 30% in a small-stocks fund. The mean monthly return rate of the market-index fund is 1.5% with a standard deviation of 0.9%. The small-stocks fund has the mean monthly return rate of 2.2% with standard deviation of 1.2%. The correlation between the two funds is 0.13. Assume normal distribution for the return rates. What is the monthly VaR at 99% level for Brian's portfolio if the portfolio value today is $100,000?

Answer:

The mean return of the portfolio is

$\mu = (0.7)(0.015) + (0.3)(0.022) = 0.0171.$ 0.5

The variance is $\sigma^2 = (0.7^2)(0.009^2) + 0.3^2\,(0.012^2) + 2(0.7)(0.3)(0.13)(0.009)(0.012) = 0.0000585468$. Thus, $\sigma = 0.0077.$ 1

So, the 99% VaR correspond to the quantile δP such that $\dfrac{\delta P - \mu}{\sigma} = -2.33$ 0.5 . Here, δP refers to the portfolio change. So, $\delta P = -2.33\sigma + \mu = -2.33(0.0077) + 0.0171 = -0.000841$ (negative indicates quantile location in the left tail of the distribution) 1

Hence, the 99% VaR is $-100{,}000(-2.33\sigma + \mu) = 100{,}000(0.000841) = \$84.10.$ 0.5 0.5

♣REQUIRED PROBLEM #4 [7 points]

Additional Problem 4

Consider the experiment of drawing (with replacement) a ball three times from a bag containing a blue ball (B) and a red ball (R).

(a) Write down the set or sample space, Ω, containing all possible outcomes. If we consider the collection of all subsets of Ω, how many sets are there in this collection? [1 pt]

(b) Suppose the probability of getting B is 3/7 and the probability of getting R is 4/7. Define or construct the probability measure for each individual element $\omega \in \Omega$. [1 pt]

(c) Write down the σ-algebra or σ-field $\{\Im_i\}$ keeping track the outcomes of each drawing for i = 0, 1, 2, 3. [2 pts]

(d) Let Ω be given as in (a) and consider the binomial asset pricing model where $S_0 = 20$, $d = 4/5$ and $u = 5/4$ so that $\{S_i\}$ is a stochastic process, i.e., S_i's are random variables for i = 0, 1, 2, 3. Find $S_2(\varpi)$, i.e., what is the function $S_2(\varpi)$? [1 pt]

(e) Consider the interval $\left[e^{\pi}\text{-}9,\ \pi^{e}+9\right]$. What is the pre-image under S_2 of this interval? Recall that S_2 is a random variable and by definition it maps Ω into $\Re$. [2 pts]

Answer:

0.5 (a) All possible outcomes of tossing a coin three times can be described by the set $\Omega = \{BBB, BBR, BRB, BRR, RBB, RBR, RRB, RRR\}$. Since there are 8 elements in Ω, there would be $2^8 = 256$ subsets of Ω contained in the collection. 0.5

(b) For the individual elements of Ω, we have

$P(BBB) = (3/7)^3 = 27/343$	$P(BBR) = (3/7)^2(4/7) = 36/343$
$P(BRB) = (3/7)^2(4/7) = 36/343$	$P(BRR) = (3/7)(4/7)^2 = 48/343$
$P(RBB) = (3/7)^2(4/7) = 36/343$	$P(RBR) = (3/7)(4/7)^2 = 48/343$
$P(RRB) = (3/7)(4/7)^2 = 48/343$	$P(RRR) = (4/7)^3 = 64/343$

0.8

Note that $\sum_{\omega \in \Omega} P\{\omega\} = 1.$ 0.2

(c) $\Im_0 = \{\phi, \Omega\}$ signifying we do not know anything yet and we note that $\phi^c = \Omega \in \Im_0$ since $\Im$ is closed under set complementation. 0.5

0.5 $\Im_1 = \{\phi, \Omega, \{BBB, BBR, BRB, BRR\}, \{RBB, RBR, RRB, RRR\}\}$, which signifies that we have either B or R on the drawing

0.5 $\Im_2 = \{\phi, \Omega, \{BBB, BBR\}, \{BRB, BRR\}, \{RBB, RBR\}, \{RRB, RRR\}$, and all sets which can be built by taking unions of these$\}$, which signifies that on the first two drawings, we have BB or BR or RB or RR.

0.5 $\Im_3 = \Im =$ The set or collection of all subsets of Ω.

Alternatively, we could simplify the notation a bit by defining

$A_B :=$ {*BBB, BBR, BRB, BRR*} = {*B* on the first drawing}
$A_R :=$ {*RBB, RBR, RRB, RRR*} = {*R* on the first drawing}
so that $\Im_1 = \{\phi, \Omega, A_B, A_R\}$.

Also, define
$A_{BB} :=$ {*BBB, BBR* } = {*BB* on the first two drawings}
$A_{BR} :=$ {*BRB, BRR* } = {*BR* on the first two drawings}
$A_{RB} :=$ {*RBB, RBR* } = {*RB* on the first two drawings}
$A_{RR} :=$ {*RRB, RRR* } = {*RR* on the first two drawings}
so that

$\Im_2 = \{\phi, \Omega, A_{BB}, A_{BR}, A_{RR}, A_B, A_R, A_{BB} \cup A_{RR}, A_{BR} \cup A_{RB}, A_{BR} \cup A_{RR}, A_{BB}{}^c, A_{BR}{}^c, A_{RB}{}^c, A_{RR}{}^c\}$

$\Im_3 = \Im =$ The set or collection of all subsets of Ω.

(d) For S_2, we have

$S_2(BBB) = S_2(BBR) = 20(5/4)^2 = 25$ 0.25

$S_2(BRB) = S_2(BRR) = S_2(RBB) =$
$S_2(RBR) = 20(5/4)(4/5) = 20(4/5)(5/4) = 20$ 0.25

$S_2(RRB) = S_2(RRR) = 20\ (4/5)^2 = 12.8.$ 0.25

Therefore, $S_2(\omega) = \begin{cases} 25 & \text{if} \quad \omega \text{ is BBB or BBR} \\ 20 & \text{if} \quad \omega \text{ is BRB, BRR, RBB or RBR} \\ 12.8 & \text{if} \quad \omega \text{ is RRB or RRR} \end{cases}$ 0.25

0.5

(e) The pre-image under S_2 of the interval $\left[e^{\pi}\text{-}9,\ \pi^{e}+9\right]$ = [14.1408, 31.4582] is defined to be

0.5 $\{\omega \in \Omega : S_2(\omega) \in [e^{\pi} - 9, \pi^e + 9]\}$ = $\{\omega \in \Omega : e^{\pi} - 9 \le S_2 \le \pi^e + 9\}$ = $A_{RR}{}^c$ since this corresponds to {BBB, BBR, BRB, BRR, RBB, RBR}.

1