ANSWERS TO ASSIGNMENT 3
Questions REQUIRED for submission

♣Required Problem #1

$S_0 = 50$, $\mu = 0.18$, $\sigma = 0.30$, $T = 2$, $r = 0.10$

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(a) The model for S_T can be rewritten as $\ln(S_T/S_0) = (\mu - \sigma^2/2)T + \sigma W_T$. 0.5

or $\ln S_T = \ln S_0 + (\mu - \sigma^2/2)T + \sigma W_T$. Since $W_T \sim N(0, T)$, $\ln S_T \sim N(\ln S_0 + (\mu-\sigma^2/2)T, \sigma^2 T)$. So, the stock price 1 distribution is lognormal. 0.5

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(b) From (a), the respective mean and standard deviation are $\ln S_0 + (\mu-\sigma^2/2)T$ and $\sigma\sqrt{T}$.

$E(S_T) = 71.67$ 0.5

Substituting the values of the parameters, we have

0.5 0.5

Mean = 4.1820 and Standard deviation = 0.4243.

MGF → $Sd(S_T) = 31.8265$ 0.5

[Alternatively, one may solve for $E[S_T]$, $E[S_T^2]$ and $Var[S_T]$ to obtain the mean and standard deviation of S_T. This can be obtained by applying the moment generating function of a normal random variable. But, given that this part is allocated only one point, this isn't really the intent of the question.]

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(c) We start with $\ln S_T \sim N(\ln S_0 + (\mu-\sigma^2/2)T, \sigma^2 T)$. Therefore, the

95% confidence interval limits for $\ln S_T$ are

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$ 0.5

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$ 0.5

So, the 95% confidence intervals for S_T are

$$e^{\ln S_0 + (\mu-\sigma^2/2)T - 1.96\sigma\sqrt{T}} \quad \text{and} \quad e^{\ln S_0 + (\mu-\sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

i.e.

$$S_0 e^{(\mu-\sigma^2/2)T - 1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0 e^{(\mu-\sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

Substituting the values, we have

50exp[(0.18-0.30²/2)2-1.96(0.30) $\sqrt{2}$] =28.52 and 0.5

50exp[(0.18-0.30²/2)2+1.96(0.30) $\sqrt{2}$] = 150.44 0.5

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(d) Under the risk neutral-world $\mu = r = 0.10$. The probability that the option will be exercised is $P(S_T > X)$, where in this case $T = 1/2$ (i.e., 6 months) and $X = 50$.

$$
\begin{aligned}
P(S_T > X) &= P(\ln S_T > \ln X) \\
&= P\left(\frac{\ln S_T - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}} > \frac{\ln X - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right) \\
&= P\left(Z > \frac{\ln(X/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right) = P\left(Z \le -\frac{\ln(X/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right) \\
&= P\left(Z \le \frac{\ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right) = \Phi(d_2),
\end{aligned}
$$

2

where $d_2 = \dfrac{\ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = 0.1296.$

Therefore, the risk-neutral probability that the option will be exercised is $P(S_T > 50)$ $\Phi(d_2) = \Phi(0.13) = 0.5517$. ✓ 1

♣Required Problem #2

Let α and Δ be the respective holdings in the money market account and stock investments.

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(a) From the principle of replication, we have the system of equations below

1.0025α

$$
\left.
\begin{aligned}
1.005\alpha + 101\Delta &= 0 \\
1.05\alpha + 99\Delta &= 1
\end{aligned}
\right\}
$$

1 $r = 0.5\%$

50.24876

Solving the above system yields $\alpha = \cancel{50.2487}6$ and $\Delta = -0.5$. (0.5) (0.5)

This means a short position in 0.5 shares and \$48.0952 must be held in the money market account. 1

50.24876 0.2488

(b) ~~The cost~~ of setting up the portfolio is at time 0 is 100(-0.5) + ~~50.2487~~

1 .e

. The put price is therefore, \$0.2488. 0.2488. 1

♣Required Problem #3

(a) With only the risky asset and cash available we can achieve pay-offs of the form

$$3a\cancel{3}[1\ \ 2\ \ 3]+3c[1\ \ 1\ \ 1] = 3[a+c\ \ 2a+c\ \ 3a+c].$$

To be able to purchase a derivative with pay-off [3 0 0], one needs to find a and c such that

$$3[a+c\ \ 2a+c\ \ 3a+c] = 3[1\ \ 0\ \ 0].\quad (*)$$ 1

3

1

However, there are no values of a and c such that equation (*) is satisfied. Thus, the pay-off [1 0 0] cannot be purchased 1 ~~since it could~~ not be made available from the combination of the bond and stock pay-offs.

1

The market is therefore **incomplete** since not all claims (e.g., 3[1 0 0]) can be replicated by the two basic securities

(b) With the risky asset, the available pay-offs are of the form $3a[1\ 2\ 3] = 3[a\ \ 2a\ \ 3a]$.

With the cash, we could have available pay-offs are of the form $3c[1\ \ 1\ \ 1]$.

With the derivative, we could have available pay-offs of the form $3d[1\ 1\ 0]$.

Thus, with the above we can buy any pay-off of the form:

$$3[a+c+d\ \ \ 2a+c+d\ \ \ 3a+c].$$

if without the expression (-0.2) each

Now, we could achieve the desired pay-off [1 0 0] by selling 1/3 unit of the asset, buying 1 unit of cash and selling 1/3 derivative: 1

$$(-1/3)[1\ \ 2\ \ 3] + (1)[1\ \ 1\ \ 1] + (-1/3)[1\ \ 1\ \ 0] = \overset{3}{[}1\ \ 0\ \ 0].$$

We can also obtain pay-off [0 1 0] using the following recipe: 1

$$(1/3)[1\ \ 2\ \ 3] + (-1)\ [1\ \ 1\ \ 1] + (2/3)[1\ \ 1\ \ 0] = [1\ \ 0\ \ 0].$$

Pay-off [0 0 1] can also be obtained through 1

$$0[1\ \ 2\ \ 3] + (1/3)[1\ \ 1\ \ 1] + (-\ 1/3)[1\ \ 1\ \ 0] = [1\ \ 0\ \ 0].$$

For the purpose of constructing attainable pay-offs, we can just as well regard the pay-offs from the newly available securities: [1 0 0], [0 1 0] and [0 0 1]. As an aside these 3 "basis" securities (similar to the concept of ***canonical basis*** in Linear Algebra) are called ***state-contingent claims*** because each pays off 1 in one and only one state and otherwise pays off 0.

To buy the **arbitrary pay-off** $[x\ \ y\ \ z]$ (meaning the market is now **complete**), it is only necessary to combine these as follows:

$$x[1\ \ 0\ \ 0] + y[0\ \ 1\ \ 0] + z[0\ \ 0\ \ 1] = [x\ y\ \ z].\qquad (**)$$

Certainly, the claim [1 0 0] is attainable since we could find x, y and z such that equation (**) holds. In particular, $x = 1$, $y = 0$ and $z = 0$. 0.5

Alternatively, one can show that the claim 3[1 0 0] is attainable using the original securities by solving the system 2 $[a+c+d \;\; 2a+c+d \;\; 3a+c] = 3[1 \;\; 0 \;\; 0]$. In this case, $a = \cancel{-1}$, $c = \cancel{+3}$ and $d = \cancel{-1}$.

(-1   3   -1) 0.5