Stats 3520b – Week of 10–14 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts were covered/reviewed:

1. When confronted with problems or questions involving options whose underlying variable is the price of futures contract or foreign exchange rate, we need to modify the calculation of the risk-neutral probability.

   In particular, the risk-neutral probability q in the implementation of the binomial option pricing approach needs to be modified to

   $$q = \frac{e^{(r-\theta)T} - d}{u - d},$$

   where $u - 1$ and $1 - d$ are the respective proportional increase and proportional decrease of the stock price. Note that the change from r to $r - \theta$ hinges on the fact that dividend is an income to the holder of the asset and therefore it decreases the stock price.

   In other words, when employing the risk-neutral probability q in the binomial option pricing model, replace θ by r for options on futures contract, replace θ by the foreign risk-free rate r_f when the underlying variable is a foreign exchange rate, and replace θ by $y - u$ when the underlying variable is a commodity price where y (in %) is the dividend yield and u (in %) is the storage cost.

   It is important to note that T refers to the time length of one period, and appropriate modification is necessary depending on the problem.

2. We started the discussion on generalising the binomial option pricing model (CRR model), i.e., extending it to the $n-$period framework with the aim of deriving an option pricing formula. Under this framework,

it is important to realise that there are two basic securities and they will replicate the contingent claim. These are:
(i) the riskless asset in the form of money market account, a savings account or a default-free bond; the riskless asset $B(t)$ has dynamics $B(t) = e^{rt}$ or $B(t) = (1+r)^t$ depending on the compounding frequency of the risk-free rate r.
(ii) risky asset, which is the stock in our framework and its price at time t is denoted by $S(t)$.

The time horizon is T and $t = 0, 1, 2, \ldots, T$. The riskless asset yields a riskless rate of return r in each time interval $[t, t+1]$. The stock price evolves according to $S(t+1) = uS(t)$ with probability p or $S(t+1) = dS(t)$ with probability $1-p$ and $0 < d < u$.

We define a RV $Z(t+1) := \dfrac{S(t+1)}{S(t)}$ for $t = 0, 1, \ldots, T-1$. We set up a probabilistic model by considering the RVs $Z(t)$ for $t = 1, \ldots, T$ defined on probability spaces $(\widetilde{\Omega}_t, \widetilde{\mathcal{F}}_t, \widetilde{P}_t)$ with
$\widetilde{\Omega}_t = \widetilde{\Omega} = \{d, u\}$
$\widetilde{\mathcal{F}}_t = \widetilde{\mathcal{F}} = \{\emptyset, \{d\}, \{u\}, \widetilde{\Omega}\}$
$\widetilde{P}_t = \widetilde{P}$ with $P(\{u\}) = p$ and $P(\{d\}) = 1-p$, $p \in (0,1)$.

The aim of the above construction is to define the probability space on which we can model the basic securities (B,S). This definition of $S(t)$ suggests that the underlying probabilistic model is the product space $(\Omega, \mathcal{F}, P)$, where Ω, $\mathcal{F}$ and P are described below:

The sample space Ω is given by $\Omega = \widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_T = \widetilde{\Omega}_T = \{d, u\}^T$ with each $\omega \in \Omega$ is a $T-$tuple $\omega = \{\widetilde{\omega}_1, \ldots, \widetilde{\omega}_T\}$ and $\widetilde{\omega}_t \in \widetilde{\Omega} = \{d, u\}$.

In our binomial model, $\widetilde{\mathcal{F}}_t = \widetilde{\mathcal{F}} = \mathcal{P}(\widetilde{\Omega}) = \{\emptyset, \{d\}, \{u\}, \widetilde{\Omega}\}$. The sigma-field or sigma-algebra $\mathcal{F}$ is therefore $\mathcal{F} = \mathcal{P}(\Omega)$.

The probability measure P is given by

$$P(\{\omega\}) = \widetilde{P}_1(\{\widetilde{\omega}_1\}) \times \widetilde{P}_2(\{\widetilde{\omega}_2\}) \times \ldots \widetilde{P}_T(\{\widetilde{\omega}_T\}) := \widetilde{P}(\{\widetilde{\omega}_1\}) \cdot \widetilde{P}(\{\widetilde{\omega}_2\}) \cdot \ldots \cdot \widetilde{P}(\{\widetilde{\omega}_T\}).$$

Remarks:
(i) The role of a product space is to model independent replication of

a random experiment.

(ii) The $Z(t)$ above are two-valued random variables, so it can be thought of as tosses of a *biased* coin. We need to build a probability space on which we can model a succession of such independent tosses.

3. Note that $Z_1, \ldots, Z_T$ defined in #2 are independent and identically distributed random variables with $\widetilde{P}(Z(t) = u) = p = 1 - \widetilde{P}(Z(t) = d)$. To model the flow of information in the market, use the obvious filtration:
$\mathcal{F}_0 = (\emptyset, \Omega)$
$\mathcal{F}_t = \sigma(Z(1), \ldots, Z(t)) = \sigma(S(1), \ldots, S(t))$
$\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$.

   The above construction emphasises that a multi-period model can be viewed as a sequence of single-period models. Indeed, in the CRR case, we use identical and independent single-period models.

   Of course, the assumption of independent and identically distributed price movements is a simplification. But, it makes the mathematics tractable.

   We showed in class the following main result for the price of a European call option under the generalised binomial-tree approach.

   Main Result: Suppose r is a risk-free rate with annual compounding. Consider a European call option with expiry T and strike X written on (one share) of the stock S. Set $\tau := T - t$. The arbitrage-free price process $c(t)$, $t = 0, 1, \ldots, T$ of the option is

$$c(t) = (1+r)^{-r\tau} \sum_{j=0}^{\tau} \left(\begin{array}{c} \tau \\ \mathrm{j} \end{array} \right) p^j (1-p)^{\tau - j} \left(S(t) u^j d^{\tau - j} - X \right)^+ ,$$

   where p is a risk-neutral probability.

   Here, j and $\tau - j$, are the respective number of times $Z(i)$ takes the two possible values, d and u. Thus, *the upper limit of the summation* τ

represents the number of time steps or subdivisions of the time horizon; the τ appearing in the discounting factor is the time to maturity of the contract (measured in years).

4. Under the $n-$period CRR model, an arbitrage-free financial market is guaranteed by the existence of an equivalent martingale measure (to be elaborated further later). It is necessary and sufficient that $d_n < 1 + r_n < u_n$.

   Here, $q_n = \dfrac{(1+r_n) - d_n}{u_n - d_n}$ and the only parameters to choose freely in the model are u_n and d_n.

5. By setting $u_n = e^{\sigma\sqrt{\delta_n}}$ and $d_n = u_n^{-1} = e^{-\sigma\sqrt{\delta_n}}$ and with risk-neutral probabilities q_n for the corresponding single-period models given by $q_n = \dfrac{1 + r_n - d_n}{u_n - d_n} = \dfrac{e^{r\delta_n} - e^{-\sigma\sqrt{\delta_n}}}{e^{\sigma\sqrt{\delta_n}} - e^{-\sigma\sqrt{\delta_n}}}$, we could write the price $c(t)$ at time t of a European call on the stock S with strike X and expiry T under the generalised binomial option pricing formula.

   Let $a_n = \min\{j \in \mathbb{Z}_0 |\ S(0)u_n^j d_n^{k_n - j} > X\}$.

   The price at time $t = 0$ in the setting of the nth CRR model is given by

$$\begin{aligned} c_n(0) &= (1+r_n)^{-k_n} \sum_{j=a_n}^{k_n} \begin{pmatrix} k_n \\ j \end{pmatrix} q_n^j (1-q_n)^{k_n - j} \left(S(0)u_n^j d_n^{k_n-j} - X\right)^+ \\ &= S(0) \left[\sum_{j=a_n}^{k_n} \begin{pmatrix} k_n \\ j \end{pmatrix} \left(\frac{q_n u_n}{1+r_n}\right)^j \left(\frac{(1-q_n)d_n}{1+r_n}\right)^{k_n-j} \right] \\ &\quad -(1+r_n)^{-k_n} X \left[\sum_{j=a_n}^{k_n} \begin{pmatrix} k_n \\ j \end{pmatrix} {q_n}^j (1-q_n)^{k_n-j} \right]. \end{aligned} \quad (1)$$

   We shall analyse what happens to equation (1) as $n \to \infty$.

6. By setting $u_n = e^{\sigma\sqrt{\delta_n}}$ and $d_n = u_n^{-1} = e^{-\sigma\sqrt{\delta_n}}$ and with risk-neutral probabilities q_n for the corresponding single-period models given by $q_n = \dfrac{1 + r_n - d_n}{u_n - d_n} = \dfrac{e^{r\delta_n} - e^{-\sigma\sqrt{\delta_n}}}{e^{\sigma\sqrt{\delta_n}} - e^{-\sigma\sqrt{\delta_n}}}$, we could write the price $c(t)$ at time t of a European call on the stock S with strike X and expiry T under the generalised binomial option pricing formula.

   Let $a_n = \min\{j \in \mathbb{Z}_0 |\ S(0)u_n^j d_n^{k_n - j} > X\}$. The price at time $t = 0$ in the setting of the nth CRR model is given by

$$
\begin{aligned}
c_n(0) &= (1 + r_n)^{-k_n} \sum_{j=a_n}^{k_n} \binom{k_n}{j} q_n^j (1 - q_n)^{k_n - j} \left(S(0)u_n^j d_n^{k_n - j} - X\right)^+ \\
&= S(0) \left[\sum_{j=a_n}^{k_n} \binom{k_n}{j} \left(\frac{q_n u_n}{1 + r_n}\right)^j \left(\frac{(1 - q_n)d_n}{1 + r_n}\right)^{k_n - j} \right] \\
&\quad -(1 + r_n)^{-k_n} X \left[\sum_{j=a_n}^{k_n} \binom{k_n}{j} {q_n}^j (1 - q_n)^{k_n - j} \right]. \qquad (2)
\end{aligned}
$$

   If we denote the binomial CDF with parameters (n, p) by $B^{n,p}(\cdot)$, we see that the second bracketed expression in equation (2) is just $\bar{B}^{k_n, q_n}(a_n) = 1 - B^{k_n, q_n}(a_n)$.

   Moreover, the first bracketed expression in equation (2) is $\bar{B}^{k_n, \widehat{q}_n}(a_n)$ with $\widehat{q}_n = \dfrac{q_n u_n}{1 + r_n}$.

   For the nth CRR model, the price of a European call at time $t = 0$ is

$$
\boxed{c_n(0) = S(0)\bar{B}^{k_n, \widehat{q}_n}(a_n) - X(1 + r)^{-k_n} \bar{B}^{k_n, q_n}(a_n)}.
$$