Stats 3520b – Week of 17-21 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts were covered/reviewed:

1. We have the following limit relation:

$$\lim_{n\to\infty} c_n(0) = c_{BS}(0)$$

with $c_{BS}(0)$ given by the **Black-Scholes-Merton formula**

$$\boxed{c_{BS}(0) = S\Phi(d_1(S,T)) - Xe^{-rT}\Phi(d_2(S,T))}, \tag{1}$$

where

$$\begin{aligned} d_1(S,T) &= \frac{\ln(S/X) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \\ d_1(S,T) &= \frac{\ln(S/X) + (r-\sigma^2/2)T}{\sigma\sqrt{T}} \\ &= d_1(S,T) - \sigma\sqrt{T}, \end{aligned} \tag{2}$$

where $\Phi(\cdot)$ is the standard normal CDF and we use $S = S(0)$.

The following remarks are in order concerning equation (2).
Remarks: It remains to show that (i) $\lim_{n\to\infty} \bar{B}^{k_n,\widehat{q}_n}(a_n) = \Phi(d_1(S,T))$ and (ii) $\lim_{n\to\infty} \bar{B}^{k_n,q_n}(a_n) = \Phi(d_2(S,T))$, where $\bar{B}^{k_n,\widehat{q}_n}(a_n)$ and $\bar{B}^{k_n,q_n}(a_n)$ were defined in the previous lecture summary. Proofs of (i) and (ii) involve the concept of convergence of distribution functions. For interested readers, refer to: Bingham and Kiesel (2000), "Risk-neutral valuation: Pricing and Hedging of Financial Derivatives", Springer-Verlag, London-Berlin-Heidelberg.

When there are dividends at rate q, the above formulae (1)-(2) needs to be modified. The formula incorporating dividends is

$$\boxed{c_{BS}(0) = Se^{-qT}\Phi(d_1(S,T)) - Xe^{-rT}\Phi(d_2(S,T))},$$

Since $\ln \frac{Se^{-qT}}{X} = \ln \frac{S}{X} - qT$, it follows that d_1 and d_2 are given by

$$\begin{aligned} d_1(S,T) &= \frac{\ln(S/X) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_1(S,T) &= \frac{\ln(S/X) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= d_1(S,T) - \sigma\sqrt{T}. \end{aligned}$$

The price p of a put option using the put-call parity can be written as $p_{BS}(0) = Xe^{-rT}\Phi(-d_2(S,T)) - Se^{-qT}\Phi(-d_1(S,T))$.

(b) The above discussion illustrates that we have derived the classical Black-Scholes-Merton valuation formula for a European call option as an *asymptotic limit* of option prices in a sequence of CRR-type models with a special choice of parameters.

(c) A straightforward analysis of the **continuous-time** Black-Scholes-Merton market model can be performed using stochastic calculus.

(d) In Advanced Financial Modelling course, the dynamics of the stock price will be driven by a stochastic process called Brownian, which is the continuous-time limit of a random walk. The stock price process $S(t)$ will be modelled as a geometric Brownian motion and the European call price can be calculated by evaluating directly the risk-neutral expectation $E[\exp(-r(T-t))(S_T - X)^+|\mathcal{F}_t]$ where $S_T = S_t \exp[(r - \sigma^2/2)(T-t) + \sigma(W(T) - W(t))]$ and $\mathcal{F}_t = \sigma(W(t))$, the $\sigma-$field generated by the Brownian motion $W(t)$.

A Brownian motion $W(t)$ is a stochastic process whose increments $W(T) - W(t)$, for $t \leq T$, are IID random variables and $W(T) - W(t) \sim N(0, T-t)$.

Note: *Remark (d) is for your additional knowledge only and intended*

only to show the further direction of SS3520B, especially if you wish to pursue SS4521G. Rest assured, no questions will be asked in the exams pertaining to Brownian motion or any aspects of the Black-Scholes-Merton formula.

2. We started providing the preliminaries of complete and incomplete markets. Essentially, we wish to explain why the absence of arbitrage implies that contingent claims that could be replicated by a trading strategy are priced under a risk-neutral probability measure.

3. It was pointed out in the lecture that the notion of martingale probability measures are important in (i) modern approach to pricing and (ii) solving portfolio-optimisation problems.

4. Let A be any process (e.g., price process, wealth process, portfolio process, etc). Let $\widetilde{A}$ be the discounted process (discounted at the risk-free rate).

   A probability measure is a *martingale measure* for the financial-market model if the discounted stock prices S_i are martingales.

5. Let the martingale measure be $Q^* = (q^*, 1 - q^*)$ of the up and down movements, respectively, such that the discounted stock price is a martingale. That is,

$$S(0) = \widetilde{S}(0) = E^{Q^*}[\widetilde{S}(1)] = q^* \frac{S(0)u}{1+r} + (1 - q^*)\frac{S(0)d}{1+r},$$

   where E^{Q^*} denotes the expectation under the probability measure Q^*. Solving for q^*, we obtain $q^* = \dfrac{(1+r) - d}{u - d}$ and $1 - q^* = \dfrac{u - (1+r)}{u - d}$.

Clearly, the assumption $d < 1 + r < u$ guarantees that these numbers q^* and $1 - q^*$ are indeed positive numbers.

6. On an intuitive level, we examined the connection between the martingale probability measure q^* and real-world measure p.

7. Within the binomial option pricing framework, the set of probabilities $\{q^*, 1 - q^*\}$ and set of probabilities $\{p, 1 - p\}$ denote the risk-neutral and real-world measures, respectively. We showed in class that $p = q^*$ when $\mu = r$, that is, when the rate of return in the real-world equals the risk-free rate.

   The above result is consistent with the fact that under the risk-neutral world there is no compensation for holding the risky asset since $\mu - r = 0$. The expected rate of return of the risky asset is equal to the risk-free rate.

8. In our market model, we have two securities: (i) riskless asset and (ii) risky asset. Consider a portfolio consisting of a bond (riskless asset) and a stock (risky asset). Let X be the wealth process with $X(0) = x$. Suppose Δ :=number of shares held in the portfolio to be invested in the stock and $x - \Delta S(0)$ is the remaining part of the portfolio to be invested in the bank at the risk-free rate r.

   Using the fact that $E^*[\widetilde{S}(1)] = \widetilde{S}(0) = S(0)$, we showed that $E^*[\widetilde{X}(1)] = \widetilde{X}(0) = X(0)$, where E^* denotes the expectation under a risk-neutral measure. This implies that the expected discounted future wealth under the risk-neutral world is equal to the initial wealth. Thus, *the discounted wealth process is a martingale under an equivalent martingale measure (EMM).*

9. We examined the concepts of self-financing and replicating portfolios.

   Definition: A portfolio *replicates* a contingent claim when the pay-off of the portfolio <u>matches</u> pay-off of the contingent claim in all possible states of the world (with probability one).

   Let $\pi(t)$ be the amount of money (not the number of shares) held in the stock investment at time t. A negative value means short-selling the stock; π can be considered as a portfolio process.

10. The formal definition of a *self-financing portfolio* will be defined in the next lecture.