# Stats 3520b – Week of 24-28 March 2014 and 31 March 2014 lecture

## SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts were covered/reviewed:

1. Write $G := G^{g,\pi}$ for the wealth process corresponding to the initial investment $g > 0$ and portfolio strategy $\pi$.

   **Definition:** A portfolio strategy $\pi$ is said to be self-financing if the amount held in the bank at time $t$ is equal to $G(t) - \pi(t)$ and $X$ satisfies

   $$\delta G = \frac{\pi}{S}\delta + \frac{G - \pi}{B}\delta B.$$

   That is the change in the wealth process is just coming from the changes in the stock and bond. No cash inflow and outflow are permitted.

2. We provided details to justify the statement: A claim $\phi$ can be replicated starting with initial wealth $\phi(0)$ if and only if $E^*[\widetilde{\phi}] = \phi(0)$. When we say that a claim can be replicated, we mean that we can find a trading strategy (i.e., the number $\Delta$) and the corresponding cost, say $V(0)$, of setting up the strategy. Eventually, it may be shown that $V(0) = \phi(0)$.

3. Two important concepts in valuation are: complete and perfect markets.

   **Definition:** A market is *complete* when we can replicate any contingent claim with the existing securities (bond and stock in this case).

   **Definition:** A market is *perfect* if the following holds: (i) complete,

(ii) arbitrage-free, and (iii) no market "frictions" (e.g., no transaction costs, no restrictions or penalties on borrowing or short-selling and no limitations of any other type on the portfolio choice).

4. To introduce the concept of state prices, we considered a general single-period model with one stock. We assume that the value of of the stock price $S(1)$ at the end of the period (say, time 1) is a random variable that can take $K$ possible values $S^1,\ S^2, \ldots, S^K$, which correspond to $K$ possible states of the world.

   A vector $\mathbf{d} = (d_1, d_2, \ldots, d_K)$ is called a *state-price vector* or a vector of *Arrow-Debreu prices*, if $d_i > 0$ for all $i = 1, 2, \ldots, K$, and we have $S(0) = \sum_{i=1}^{K} d_i S^i$.

   The above representation for $S(0)$ means that the stock price today is obtained as a weighted average of its future values with the weights being the state prices.

5. We assume that there is a risk-free asset $B_0$ such that $B_0(1) = 1 + r$, where $r > 0$ is constant. If we assume a bond paying 1 in the future then the analogue of $S(0) = \sum_{i=1}^{K} d_i S^i$ would be

$$\frac{1}{1+r} = \sum_{i=1}^{K} d_i.$$

   That is, $S^i = 1$, $\forall\ i$.

6. Recall the state-price vector or vector of Arrow-Debreu prices $\mathbf{d} = (d_1, d_2, \ldots, d_K)$, where each $d_i$ is positive.

Write $q_i^* = \dfrac{d_i}{\sum_{j=1}^K d_j} = (1+r)d_i$ using the fact that $\dfrac{1}{1+r} = \displaystyle\sum_{i=1}^K d_i$. Here, $r$ is the risk-free rate.

We showed in class that the stock price process $S$ satisfies the relation

$$S(0) = \sum_{i=1}^K q_i^* \frac{S^i}{1+r} = E^*[\widetilde{S}(1)],$$

where $E^*$ is the expected value under the probabilities $q_i^*$, $\widetilde{S}$ is the discounted price of the stock and $S^i$ is the stock price in state $i$.

The above shows that the discounted stock prices are martingales under $q_i^*$. Note that $q_i^*$ is a risk-neutral probability measure or **an equivalent martingale measure**.

7. Clearly, as demonstrated in #6 *the existence of state prices is equivalent to the existence of risk-neutral probability measure.*

8. **Fundamental Theorem of Asset Pricing:** Consider a discrete-time financial-market model with finitely many possible outcomes.
(i) If there exists a martingale measure with positive probabilities then the market is arbitrage-free. Conversely, if the market is arbitrage-free then there exists a martingale measure with positive probabilities.
(ii) If there exists a unique martingale measure with positive probabilities then the market is complete and arbitrage-free. Conversely, if the market is complete and arbitrage-free then there exists a unique martingale measure with positive probabilities.

9. A summary of pricing by no-arbitrage and risk-neutral valuation principle was presented in the lecture.

**Pricing by no-arbitrage:** If a claim $\phi$ can be replicated then the price of the claim has to be equal to its replication cost $\phi(0)$ (as further argued below); otherwise, there is an arbitrage opportunity. That is, $\phi(0)$ is the no-arbitrage price.

**Pricing by risk-neutral approach:** Denote by $G^\phi$ the replicating wealth process so that $G^\phi(T) = \phi$ and $\phi(0) = G^\phi(0)$ is the price of $\phi$ (which must also be equal to $V(0)$ in the context of #2).

We demonstrated that a discounted wealth process is a martingale under the risk-neutral measure $Q^*$. This implies that $G^\phi(0) = E^* \left[\widetilde{G}^\phi(T)\right]$ or equivalently, $\phi(0) = E^* \left[\widetilde{G}(T)\right] = E^* \left[e^{-rT} G(T)\right]$. Hence, the price of a claim $\phi$ is clearly equal to the risk-neutral expected value of its discounted pay-off.

## SWAPS

10. We considered swaps, which are private agreements between two companies to exchange a set of cashflows in the future according to a prearranged formula. This financial instrument can be regarded as portfolios of forward contracts.

11. For a plain vanilla interest rate swaps (IRS), one party B, agrees to pay the other party A cashflows at a predetermined fixed rate on a notional principal for a fixed period of time. In return, party A agrees to pay party B cashflows equal to interest at a floating rate on the same notional principal for the same period of time.

12. Parties A and B enter into such an agreement due to the so-called comparative advantage. Some companies appear to have a comparative advantage in fixed-rate markets whilst other companies have a comparative advantage in floating-rate markets. A swap has the effect of transforming a fixed rate loan into a floating rate loan and vice-versa.

13. LIBOR, which stands for London Interbank Offer rate, is the floating rate in many IRS agreements. This is similar to the prime rate, which is the reference rate in the domestic financial market.

14. To design a swap agreement, it is essential to determine first the potential gain=$|a-b|$ where $a$ =positive difference between the interest rates offered to the 2 companies in the fixed rate markets and $b$ =positive difference between the interest rates offered to the 2 companies in the floating rate markets.

15. Usually, two companies do not get in touch with each other to arrange a swap. They each deal with a financial intermediary such as a bank. This means that the total potential gain has to be split 3 ways between the two companies and the financial institution.

16. Principal payments are NOT exchanged in an IRS. This is because the dollar value of the principal remains the same throughout the contract for both the floating-rate loan and the fixed-rate loan.

17. Currency swaps involve the
**exchange of principal and fixed interest rate payments** on a loan in one currency for principal and fixed-rate payments on an approximately equivalent loan in another currency.

    The principal amounts in each currency are usually exchanged at the beginning and at the end of the life of the swap. In particular, principal payments flow in the opposite direction to the arrows at the start of the swap and in the same direction as the arrows at the end of the life

of the swap.

Currency swaps are also motivated by the comparative advantage argument.

18. CRITICISMS OF COMPARATIVE ADVANTAGE ARGUMENTS: The reason why spread differentials appear to continue to exist may in part be due to the contracts available to companies in fixed- and floating-rate markets. The rates available to two companies in the fixed rate markets are likely to be the rates at which the companies can issue fixed rate bonds.

    If the creditworthiness of both companies declines, the lender has the option of increasing the spread over LIBOR that is charged. However, providers of fixed rate financing do not have the option to change the terms of the loan. In the end, when the rates are revised, it may happen that a company's expected total borrowing rate if it enters into a swap may be greater than the rate a company could get by going directly to the outside lenders.

19. Swaps are usually arranged by financial institutions. Ideally, in order to eliminate interest or exchange rate risk, a financial institution would like to enter into swap agreements with two parties at the same time. In practice, financial institutions frequently **warehouse** swaps. This means that they enter into a swap agreement with one party and then hedge their risk on a day-to-day basis whilst they attempt to find a party wanting to take the opposite position.