Statistical Sciences 4521G/9521B
Winter 2014

Assignment No.1 Due: 11:30, 30 January 2014, WSC 248

GUIDELINES ON SUBMITTING ASSIGNMENTS

- Only problems marked with a $\star$ are required for submission, and there are four of them in this problem set. The remaining problems are practice exercises.

- Your assignment paper must include the Marking Scheme as a cover page. This marking scheme can be downloaded from the course website. **Failure to follow this instruction can result to a 2-point deduction on your assignment mark.**

- YOU MUST WRITE YOUR OWN WORK IN YOUR OWN WORDS, using full sentences and proper English grammar. It is your responsibility to familiarise yourself with the provisions of the University Regulation concerning academic integrity and honesty. **Any behaviour that can potentially lead to plagiarism, cheating and copying from/sharing with another student answers in an assignment or exams is a serious offence and carries with it severe penalty.** Do not take this warning lightly; academic penalties have dire consequences on your future studies and career.

- Do not submit your rough work! Do the problem set and then re-write it at least once - neatly, with adequate amount of clear explanation. The rewriting stage is the most important one for finding errors in one's work, and it will also deepen your understanding of the subject matter. **Assignments are marked for both technical correctness and elegance of presentation.**

- Bear in mind to include a sufficient amount of explanation about your work so that any marker does not have to guess what you mean. The grader of your work will determine if you understand what you are writing, not merely that you reach the particular correct answer.

- On questions where a computer output is required or deemed necessary, include the output in the text of your answer at the appropriate locations - do not put it all in a bunch at the end of your assignment. Unless, you are instructed to submit your work in a CD or disc, you are expected to hand in a PRINTED COPY.

Do as indicated. ENJOY!

1. {Hull 8th ed - Problem 13.3}
   A company's cash position (in million of dollars) follows a generalised Wiener process with a drift of 0.5 per quarter and variance rate of 4.0 per quarter. How high does the company's initial position have to be for the company to have a less than 5% chance of negative cash position at the end of one year?

2. { Hull 8th ed - Problem 13.4 }
   Variables X_1 and X_2 follow generalised Wiener proceses with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. What process does $X_1 + X_2$ follow if:
   (a) The changes in X_1 and X_2 in any short time interval are uncorrelated?
   (b) There is a correlation ρ between the changes in X_1 and X_2 in any short time interval?

3. {Hull 8th ed - Problem 13.5}
   Consider a variable S that follows the process $dS_t = \mu dt + \sigma dW_t$. For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year 6?

4. {Hull 8th ed - Problem 13.10}
   Suppose that a stock price S_t follows a geometric Brownian motion. Show that if $n \in \mathbb{R}$, the process S_t^n also follows a geometric Brownian motion.

5. {Hull 8th ed - Problem 13.11}
   Suppose that x_t is the yield to maturity with continuous compounding

on a zero-coupon bond that pays \$1 at time T. Assume that x_t is stochastic and follows the dynamics

$$dx_t = a(x_0 - x_t)dt + s x_t dW_t$$

where a, x_0 and s are positive constants and W_t is a Brownian motion. Determine the process followed by the bond price.

6. { Hull 8th ed - Problem 14.2}
   The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

7. {Hull 8th ed - Problem 14.3}
   Explain the principle of risk-neutral valuation.

8. {Hull 8th ed - Problem 14.9}
   Using the notation in the lecture, prove that if S_t has a lognormal dynamics then a 95% confidence interval for S_T is

$$\left(S_t e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) - 1.96\sigma\sqrt{T-t}},\ S_t e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + 1.96\sigma\sqrt{T-t}} \right).$$

9. {Hull 8th ed - Problem 14.12}
   Consider a derivative that pays off S_T^n at time T, where S_T is the stock price at that time. When the stock price follows a geometric Brownian motion, it can be shown that its price at time t ($t \leq T$) has the form $h(t,T)S^n$ where S is the stock price at time t and h is a function only of t and T.
   a. By substituting into the Black-Scholes partial differential equation, derive an ordinary differential equation satisfied by $h(t,T)$.
   b. What is the boundary condition for the differential equation for

$h(t, T)$?

c. Show that $h(t, T) = \exp[0.5\sigma^2 n(n-1) + r(n-1)(T-t)]$ where r is the risk-free interest rate and σ is the stock price volatility.

10. **Required Problem #1**

    * {Exercise given in lecture}

    Let $Z \sim N(0, 1)$ and $X_t = \sqrt{t}Z$. Find the distribution of X_t. Is X_t a Brownian motion? Provide support to your answer. [3 points]

11. {Exercise given in lecture}

    Suppose $\{W_t\}$ and $\{\widetilde{W}_t\}$ are two independent standard Brownian motions and ρ is a constant between -1 and 1. What is the distribution of $X_t := \rho W_t + \sqrt{1-\rho^2}\widetilde{W}_t$? Is X_t a Brownian motion? Justify your answer.

12. **Required Problem #2**

    * Suppose $\{W_t : t \geq 0\}$ is a standard Brownian motion. Using the definition of a martingale, is W_t^2 a martingale? If it is not a martingale, introduce (an) adjustment term(s) to make it a martingale. Hint: Consider $E[W_t^2 - W_s^2 | \mathcal{F}_s]$ and note that $E[(W_t - W_s)^2 | \mathcal{F}_s] = t - s$, where $\mathcal{F}_s$ is the filtration generated by W_t up to time s ($s \leq t$). [3 points]

13. **Required Problem #3**

    *{Item (a) is an exercise given in the lecture whilst (b) is new.}*

    Let W_t be a standard Brownian motion.

    (a) Assume that the price process S_t follows the dynamics $dS_t = \mu dt + \sigma dW_t$. Show that if S_t initially starts at S_0 then for all $\sigma > 0$ and $T > 0$ there is a positive probability that S_T is negative. [2 points]

    (b) For some constant C, show that the process defined by

    $$X_t = \sinh(C + t + W_t),$$

where W_t is a Brownian motion, is a solution of the stochastic differential equation (SDE)

$$dX_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right) dt + \left(\sqrt{1+X_t^2}\right) dW_t.$$

If $X_0 = 0$ is the given initial value to the process with the above SDE, what is the numerical value of C, if it exists?
N.B. *If you are not familiar with hyperbolic functions or have already forgotten them and their derivatives, consult a Calculus textbook.* [4 points]

14. **Required Problem #4**
$\star$ {Exercise given/will be given in lecture}
Let $X \sim N(m,\ \sigma^2)$ and $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{s^2}{2}} ds$. Suppose $c > 0$.
a. Show $P\left(e^X > c\right) = \Phi\left(\frac{m - \ln c}{\sigma}\right)$. [4 points]
b. Show $E\left[e^X I_{\{X>c\}}\right] = e^{m+\frac{\sigma^2}{2}} \Phi\left(\frac{m + \sigma^2 - c}{\sigma}\right)$ where I is an indicator function. [4 points]
N.B. *Results in 14a and 14b are lemmas (proofs not provided in class) that could be employed in the derivation of the Black-Scholes-Merton formula in the lecture.*

15. {Hull 8th ed - Problem 14.18}
Show that the Black-Scholes formulae for call and put options satisfy the put-call parity.

16. {Hull 8th ed - Problem 14.22}
Show that the probability that a European call option will be exercised in a risk-neutral world, with the notation introduced in the lecture, is $\Phi(d_2)$. What is the expression for the value of a derivative that pays

off \$100 if the price of a stock at time T is greater than X?

17. {Hull 8th ed - Problem 14.23}
    Show that S^{-2r/σ^2} could be the price of a traded security.

18. The price of a European call that expires six months and has a strike price of \$30 is \$2. The underlying stock price is \$29 and a dividend yield of \$0.50 and is expected in two months and again five months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put that expires in six months and has a strike price of \$30?

19. {Hull 8th ed - Problem 16.2}
    "Once we know how to value options on a stock paying a dividend yield, we know how to value options on stock indices, currencies and futures." Explain this statement.

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