Statistical Sciences 4521G/9521B
Winter 2014

Assignment No.1

SOLUTIONS TO NON-REQUIRED PROBLEMS

1. {Hull 8th ed - Problem 13.3}
A company's cash position (in million of dollars) follows a generalised Wiener process with a drift of 0.5 per quarter and variance rate of 4.0 per quarter. How high does the company's initial position have to be for the company to have a less than 5% chance of negative cash position at the end of one year?

SOLUTION: Suppose the company's initial cash position is x. The probability distribution of the cash position at the end of one year is $N(x + 4(0.5), 4(4)) = N(x + 2.0, 16)$. Thus, the probability of a negative cash position at the end of one year is $\Phi\left(-\frac{x+2.0}{4}\right)$ where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable. We have to find x such that $\Phi\left(-\frac{x+2.0}{4}\right) = 0.05$. From the normal distribution table, this happens when $-\frac{x+2.0}{4} = -1.6449$, i.e., $x = 4.5796$. The initial cash position must therefore be \$4.58 million.

2. { Hull 8th ed - Problem 13.4 }
Variables X_1 and X_2 follow generalised Wiener proceses with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. What process does $X_1 + X_2$ follow if:
(a) The changes in X_1 and X_2 in any short time interval are uncorrelated?
(b) There is a correlation ρ between the changes in X_1 and X_2 in any short time interval?

SOLUTION: (a) Suppose that X_1 and X_2 equal a_1 and a_2 initially. After a time period of length T, X_1 has the probability distribution $N(a_1+$

$\mu_1 T, \sigma_1^2 T)$ and X_2 has a probability distribution $N(a_2 + \mu_2 T, \sigma_2^2 T)$. From the property of sums of independent normally distributed variables, $X_1 + X_2$ has the probability distribution $N(a_1 + \mu_1 T + a_2 + \mu_2 T, \sigma_1^2 T + \sigma_2^2 T)$, i.e., $N\left(a_1 + a_2 + (\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2)T\right)$. This shows that $X_1 + X_2$ follows a generalised Wiener process with drift rate $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

(b) In this case, since μ_1, μ_2, σ_1, σ_2 and ρ are all constant, this implies that the distribution of the sum $X_1 + X_2$ in a longer period of time T is

$$N\left(a_1 + a_2 + (\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)T\right).$$

The variable $X_1 + X_2$, therefore, follows a generalised Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$.

3. {Hull 8th ed - Problem 13.5}
   Consider a variable S that follows the process $dS_t = \mu dt + \sigma dW_t$. For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year 6?

   SOLUTION: The change in S during the first three years has the probability distribution $N(2(3), 9(3)) = N(6, 27)$. The change during the next three years has the probability distribution $N(3(3), 16(3)) = N(9, 48)$.

   The change during the six years is the sum of a variable with probability distribution $N(6, 27)$ and a variable with probability distribution $N(9, 48)$. The probability distribution of the change is therefore $N(15, 75)$. Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is $N(20, 75)$.

4. {Hull 8th ed - Problem 13.10}
Suppose that a stock price S_t follows a geometric Brownian motion. Show that if $n \in I\!R$, the process S_t^n also follows a geometric Brownian motion.

SOLUTION: If $G(S,t) = S^n$ then $\dfrac{\partial G}{\partial t} = 0$, $\dfrac{\partial G}{\partial S} = nS^{n-1}$, and $\dfrac{\partial^2 G}{\partial S^2} = n(n-1)S^{n-2}$. Using Itô's lemma,

$$dG_t = \left(\mu n G + \frac{1}{2}n(n-1)\sigma^2 G\right) dt + \sigma n G dW_t.$$

This shows that $G = S^n$ follows a geometric Brownian motion where the expected return is $\mu n + \dfrac{1}{2}n(n-1)\sigma^2$ and the volatility is $n\sigma$. The stock price S has an expected return of μ and the expected value of S_T is $S_0 e^{\mu T}$. The expected value of S_T^n is $S_0^n e^{\left(\mu n + \frac{1}{2}n(n-1)\sigma^2\right)T}$.

5. {Hull 8th ed - Problem 13.11}
Suppose that x_t is the yield to maturity with continuous compounding on a zero-coupon bond that pays \$1 at time T. Assume that x_t is stochastic and follows the dynamics

$$dx_t = a(x_0 - x_t)dt + sx_t dW_t$$

where a, x_0 and s are positive constants and W_t is a Brownian motion. Determine the process followed by the bond price.

SOLUTION: The process followed by B, the bond price, is from Itô's formula given by

$$dB_t = \left(\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}\right) dt + \frac{\partial B}{\partial x} sx dW_t.$$

Since $B = e^{-x(T-t)}$ the required partial derivatives are:

$$\begin{aligned}\frac{\partial B}{\partial t} &= xe^{-x(T-t)} = xB \\ \frac{\partial B}{\partial x} &= -(T-t)e^{-x(T-t)} = -(T-t)B \\ \frac{\partial^2 B}{\partial x^2} &= (T-t)^2 e^{-x(T-t)} = (T-t)^2 B.\end{aligned}$$

Hence,

$$dB_t = \left(-a(x_0 - x)(T-t) + x + \frac{1}{2}s^2x^2(T-t)^2\right)Bdt - sx(T-t)BdW_t.$$

6. { Hull 8th ed - Problem 14.2}
The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

SOLUTION: The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and assuming 252 trading days in one year, $\Delta t = \frac{1}{252} = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

7. {Hull 8th ed - Problem 14.3}
Explain the principle of risk-neutral valuation.

SOLUTION: The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in the risk-neutral world as they do in the real world. We may therefore assume that the world is risk-neutral for the purpose of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to the risk-free rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

8. {Hull 8th ed - Problem 14.9}
Using the notation in the lecture, prove that if S_t has a lognormal dynamics then a 95% confidence interval for S_T is

$$\left(S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)-1.96\sigma\sqrt{T-t}},\ S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)+1.96\sigma\sqrt{T-t}} \right).$$

SOLUTION: From the lecture it was shown that
$\ln S_T \sim N\left(\ln S_t + \left(\mu - \dfrac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right)$. The respective left and right endpoints of a 95% confidence interval for $\ln S_T$ are therefore

$$\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t) - 1.96\sigma\sqrt{T-t}$$

and

$$\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)T + 1.96\sigma\sqrt{T-t}.$$

The respective left and right endpoints of a 95% confidence interval for S_T are therefore

$$e^{\ln S_t+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)-1.96\sigma\sqrt{T-t}}$$

and

$$e^{\ln S_t+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)+1.96\sigma\sqrt{T-t}}.$$

That is,

$$S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)-1.96\sigma\sqrt{T-t}}$$

and

$$S_t e^{\left(\mu-\frac{\sigma^2}{2}\right)(T-t)+1.96\sigma\sqrt{T-t}}.$$

9. {Hull 8th ed - Problem 14.12}
Consider a derivative that pays off S_T^n at time T, where S_T is the stock price at that time. When the stock price follows a geometric Brownian motion, it can be shown that its price at time t ($t \leq T$) has the form $h(t,T)S^n$ where S is the stock price at time t and h is a function only of t and T.
a. By substituting into the Black-Scholes partial differential equation, derive an ordinary differential equation satisfied by $h(t,T)$.
b. What is the boundary condition for the differential equation for $h(t,T)$?
c. Show that $h(t,T) = \exp[0.5\sigma^2 n(n-1) + r(n-1)(T-t)]$ where r is the risk-free interest rate and σ is the stock price volatility.

SOLUTION: (a) If $G(S,t) = h(t,T)S^n$ then $\dfrac{\partial G}{\partial t} = h_t S^n$, $\dfrac{\partial G}{\partial S} = hnS^{n-1}$ and $\dfrac{\partial^2 G}{\partial S^2} = hn(n-1)S^{n-2}$ where $h_t = \frac{\partial h}{\partial t}$. Substituting into the Black-Scholes differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh.$$

(b) The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$.

(c) The equation $h(t,T) = e^{\left[0.5\sigma^2 n(n-1)+r(n-1)\right](T-t)}$ satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a) directly. The differential equation (DE) can be re-written as

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1).$$

The solution to this DE is

$$\ln h = \left(-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)\right) t + k,$$

where k is a constant. Since $\ln h = 0$ when $t = T$ it follows that

$$k = \left(r(n-1) + \frac{1}{2}\sigma^2 n(n-1) \right) T$$

so that

$$\ln h = \left(r(n-1) + \frac{1}{2}\sigma^2 n(n-1) \right) (T-t)$$

or

$$h(t,T) = e^{\left[0.5\sigma^2 n(n-1) + r(n-1)\right](T-t)}.$$

10. **Required Problem #1**

*\{Exercise given in lecture\}

Let $Z \sim N(0,1)$ and $X_t = \sqrt{t}Z$. Find the distribution of X_t. Is X_t a Brownian motion? Provide support to your answer. [3 points]

SOLUTION: Solution to a required problem is provided in a different sheet, and will be posted after the assignment due date.

11. \{Exercise given in lecture\}

Suppose $\{W_t\}$ and $\{\widetilde{W}_t\}$ are two independent standard Brownian motions and ρ is a constant between -1 and 1. What is the distribution of $X_t := \rho W_t + \sqrt{1-\rho^2}\widetilde{W}_t$? Is X_t a Brownian motion? Justify your answer.

Outline of SOLUTION: To determine if X_t is a BM, we need to check that all condiitons in the definition is satisfied.

(i) $X_0 = 0$ since both W_0 and $\widetilde{W}_0$ are zero.

(ii) X_t is continuous since it is a linear combination of W_t and $\widetilde{W}_t$, which are both continuous standard BMs.

(iii) You should verify that for $s \le t$, $E[X_t - X_s] = 0$. Also, due to independence between W_t and $\widetilde{W}_t$, it can be shown easily that

$\mathrm{Var}[X_t - X_s] = t - s$. Hence, $X_t - X_s \sim N(0, t-s)$. That is, increments $X_t - X_s$ are *stationary* (i.e., variance depends only on the interval $t - s$.)
(iv) Note that $W_t - W_s$ is independent of $\mathcal{F}_s$ and $\widetilde{W}_t - \widetilde{W}_s$ is independent of $\mathcal{F}_s$. You can may then argue that $X_t - X_s$ (verify calculation) is independent of $\mathcal{F}_s$.

12. **Required Problem #2**
* Suppose $\{W_t : t \geq 0\}$ is a standard Brownian motion. Using the definition of a martingale, is W_t^2 a martingale? If it is not a martingale, introduce (an) adjustment term(s) to make it a martingale. Hint: Consider $E[W_t^2 - W_s^2 | \mathcal{F}_s]$ and note that $E[(W_t - W_s)^2 | \mathcal{F}_s] = t - s$, where $\mathcal{F}_s$ is the filtration generated by W_t up to time s $(s \leq t)$. [3 points]

SOLUTION: Solution to a required problem is provided in a different sheet, and will be posted after the assignment due date.

13. **Required Problem #3**
$^\star\{$Item (a) is an exercise given in the lecture whilst (b) is new.$\}^\star$
Let W_t be a standard Brownian motion.
(a) Assume that the price process S_t follows the dynamics $dS_t = \mu dt + \sigma dW_t$. Show that if S_t initially starts at S_0 then for all $\sigma > 0$ and $T > 0$ there is a positive probability that S_T is negative. [2 points]

(b) For some constant C, show that the process defined by

$$X_t = \sinh(C + t + W_t),$$

where W_t is a Brownian motion, is a solution of the stochastic differential equation (SDE)

$$dX_t = \left(\sqrt{1 + X_t^2} + \frac{1}{2} X_t\right) dt + \left(\sqrt{1 + X_t^2}\right) dW_t.$$

If $X_0 = 0$ is the given initial value to the process with the above SDE, what is the numerical value of C, if it exists?
N.B. *If you are not familiar with hyperbolic functions or have already forgotten them and their derivatives, consult a Calculus textbook.* [4 points]

SOLUTION: Solution to a required problem is provided in a different sheet, and will be posted after the assignment due date.

14. **Required Problem #4**
* {Exercise given/will be given in lecture}
Let $X \sim N(m,\ \sigma^2)$ and $\Phi(y) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{y} e^{-\frac{s^2}{2}} ds$. Suppose $c > 0$.
a. Show $P\left(e^X > c\right) = \Phi\left(\dfrac{m - \ln c}{\sigma}\right)$. [4 points]
b. Show $E\left[e^X I_{\{X>c\}}\right] = e^{m+\frac{\sigma^2}{2}} \Phi\left(\dfrac{m + \sigma^2 - c}{\sigma}\right)$ where I is an indicator function. [4 points]
N.B. *Results in 14a and 14b are lemmas (proofs not provided in class) that could be employed in the derivation of the Black-Scholes-Merton formula in the lecture.*

SOLUTION: Solution to a required problem is provided in a different sheet, and will be posted after the assignment due date.

15. {Hull 8th ed - Problem 14.18}
Show that the Black-Scholes formulae for call and put options satisfy the put-call parity.

SOLUTION: From the Black-Scholes equations

$$p_t + S_t = Xe^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1) + S_t.$$

Since $1 - \Phi(-d_1) = \Phi(d_1)$ this is

$$Xe^{-r(T-t)}\Phi(-d_2) + S_t\Phi(d_1).$$

Also,

$$c_t + Xe^{-r(T-t)} = S_t\Phi(d_1) - Xe^{-r(T-t)}\Phi(d_2) + Xe^{-r(T-t)}.$$

Since $1 - \Phi(d_2) = \Phi(-d_2)$, this is also

$$Xe^{-r(T-t)}\Phi(-d_2) + S_t\Phi(d_1).$$

16. {Hull 8th ed - Problem 14.22}
Show that the probability that a European call option will be exercised in a risk-neutral world, with the notation introduced in the lecture, is $\Phi(d_2)$. What is the expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than X?

SOLUTION: The probability that the call option will be exercised is the probability that $S_T > X$ where S_T is the stock price at time T. In a risk-neutral world,

$$\ln S_T \sim N(\ln S_t + (r - \sigma^2/2)(T-t), \sigma^2(T-t)).$$

The probability that $S_T > X$ is the same as the probability that $\ln S_T > \ln X$. This is

$$1-\Phi\left(\frac{\ln X - \ln S_t - (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) = \Phi\left(\frac{\ln(S_t/X) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) = \Phi(d_2).$$

17. {Hull 8th ed - Problem 14.23}
Show that S^{-2r/σ^2} could be the price of a traded security.

SOLUTION: If $f = S^{-2r/\sigma^2}$ then

$$\begin{aligned}
\frac{\partial f}{\partial S} &= -\frac{2r}{\sigma^2} S^{-2r/\sigma^2 - 1} \\
\frac{\partial^2 f}{\partial S^2} &= \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2} + 1\right) S^{-2r/\sigma^2 - 2} \\
\frac{\partial f}{\partial t} &= 0. \\
\text{Hence, } \frac{\partial f}{\partial t} &+ rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rS^{-2r/\sigma^2} = rf.
\end{aligned}$$

This shows that the Black-Scholes equation is satisfied. S^{-2r/σ^2} could therefore be the price of a traded security.

18. The price of a European call that expires six months and has a strike price of \$30 is \$2. The underlying stock price is \$29 and a dividend yield of \$0.50 and is expected in two months and again five months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put that expires in six months and has a strike price of \$30?

SOLUTION: If D represents the present value of the dividends, then we have the put-call parity $c_t + Xe^{-r(T-t)} + D = p_t + S_t$ or $p_t = c_t + Xe^{-r(T-t)} + D - S_t$.

In this case, $p_t = 2 + 30e^{(-0.1)(6/12)} + 0.5e^{(-0.1)(2/12)} + 0.5e^{(-0.1)(5/12)} - 29 =$\$2.51.

19. {Hull 8th ed - Problem 16.2}
"Once we know how to value options on a stock paying a dividend

yield, we know how to value options on stock indices, currencies and futures." Explain this statement.

A stock index is analogous to a stock paying a continuous dividend yield, the dividend yield being the dividend yield on the index. A currency is analogous to a stock paying a continuous dividend yield, the dividend yield being the foreign risk-free interest rate. A futures contract is analogous to a stock paying a continuous dividend yield, the dividend yield being the domestic risk-free interest rate.

$\sim\sim\sim$ **E N D** $\sim\sim\sim$