SS 4521 G/ FM 9521B

SOLUTIONS TO ASSIGNMENT #1

WINTER 2014

Problems Required for Submission

3 POINTS

REQUIRED PROBLEM #1: $X_t = \sqrt{t} Z$ starts at 0 with value 0, and it is a continuous-time process as t is a continuous function.

2

We consider the increment $X_{t+s} - X_s$, for some $s > 0$.

$$X_{t+s} - X_s = \sqrt{t+s}\, Z - \sqrt{s}\, Z = \left(\sqrt{t+s} - \sqrt{s}\right) Z$$

$E[X_{t+s} - X_s] = 0$. The variance is given by

$$Var[X_{t+s} - X_s] = \left(\sqrt{t+s} - \sqrt{s}\right)^2 Var[Z]$$

$$= (t+s) - 2\sqrt{(t+s)s} + s$$

$$= t - 2\sqrt{s^2(t/s + 1)} + 2s = t - 2\left(s\sqrt{t/s + 1} - 1\right),$$

which is not t.

Therefore, $X_{t+s} - X_s$ is normal but does not have the correct distribution for it be a BM.

The result for the variance demonstrate that 0.5 it is not stationary (variance not a function of the time increment t) and not 0.5 independent of s (given that the dist of $X_{t+s} - X_s$ has a variance containing s).

3 POINTS

REQUIRED PROBLEM #2: If W_t^2 is a martingale then $E[W_t^2 / \mathcal{F}_s] = W_s^2$ for a BM W_t defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

We claim that $E[W_t^2 / \mathcal{F}_s] > W_s^2$. Hence, W_t^2 is not a martingale.

Proof of Claim: Consider $E[W_t^2 - W_s^2 / \mathcal{F}_s]$. for $0 < s < t$.

$W_t^2 - W_s^2 = (W_t - W_s)^2 + 2W_s(W_t - W_s)$. Thus,

$$E[(W_t^2 - W_s^2) | \mathcal{F}_s] = E[(W_t - W_s)^2 | \mathcal{F}_s] + 2E[W_s(W_t - W_s) | \mathcal{F}_s]$$

$= (t-s) + 2(0)$ because W_s and $W_t - W_s$ are independent

$= t-s$.

1.5

We showed that $E[W_t^2 - W_s^2 | \mathcal{F}_s] = t-s$, which implies that

$$E[W_t^2 | \mathcal{F}_s] = W_s^2 + \underbrace{(t-s)}_{>0}$$

Since $t > s$ from the assumption above.

Clearly,

$E[W_t^2 | \mathcal{F}_s] > W_s^2$, and

W_t^2 is an increasing process on average. (Such process is called a submartingale.)

W_t^2 is NOT a martingale.

0.5

But, the relation $E[W_t^2 - W_s^2 / \mathcal{F}_s] = t-s$ is equivalent to $E[W_t^2 - t | \mathcal{F}_s] = W_s^2 - s$.

So, the process $W_t^2 - t$ is a martingale.

Clearly, $-t$ is the adjustment term need so that the martingale condition is satisfied.

1

REQUIRED PROBLEM #3 $\boxed{\text{2 POINTS}}$

(a) $S_t = \mu dt + \sigma dW_t$. This means

$$S_T = S_0 + \mu T + \sigma W_T \sim N(S_0 + \mu T, \sigma^2 T).$$

(1)
$$P(S_T < 0) = P\left(\frac{S_T - (S_0 + \mu T)}{\sigma\sqrt{T}} < \frac{-S_0 - \mu T}{\sigma\sqrt{T}}\right)$$

(1)
$$= P\left(Z < \frac{-S_0 - \mu T}{\sigma\sqrt{T}}\right),$$ which is positive

for constants S_0, μ and $\sigma > 0$, $Z \sim N(0,1)$.

$\boxed{\text{4 POINTS}}$

(b) Consider $f(t, y) = \sinh(c + t + y)$

(1)
$$\frac{\partial f}{\partial t} = \cosh(c + t + y), \quad \frac{\partial f}{\partial y} = \cosh(c + t + y)$$

$$\frac{\partial^2 f}{\partial y^2} = \sinh(c + t + y)$$

(1)
$$dX_t = df(t, W_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial y} dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dW_t)^2$$

$$= \cosh(c + t + W_t)\, dt + \cosh(c + t + W_t)\, dW_t + \frac{1}{2}\sinh(c + t + W_t)\, dt$$

$$= \left(\cosh(c + t + W_t) + \frac{1}{2}\sinh(c + t + W_t)\right) dt + \cosh(c + t + W_t)\, dW_t.$$

Using the identity that

$\cosh^2\theta - \sinh^2\theta = 1$, i.e., $\cosh\theta = \sqrt{1 + \sinh^2\theta}$, 1

we have

$$dX_t = \left(\sqrt{1+\sinh^2(C+t+W_t)} + \frac{1}{2}\sinh(C+t+W_t)\right)dt + \sqrt{1+\sinh^2(C+t+W_t)}\,dW_t.$$

$$= \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}\,dW_t.$$

1

If $X_0 = 0$, we have

$$X_0 = \sinh(C + 0 + \overset{0}{\overset{\shortparallel}{W_0}}) = \sinh C$$

$\Rightarrow$ $C = \sinh^{-1}(0)$; and wish a numerical value.

But note that $\sinh = \dfrac{e^C - e^{-C}}{2} = 0$

$$\Rightarrow e^C = e^{-C} \Rightarrow e^{2C} = e^0$$

$$\Rightarrow 2C = 0 \Rightarrow C = 0.$$

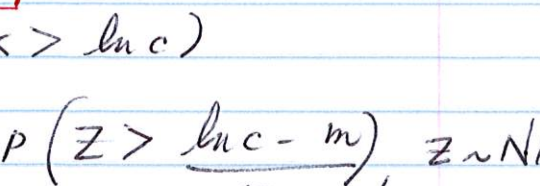
REQUIRED PROBLEM #4: $\boxed{\text{4 POINTS}}$

(a) $P(e^X > c) = P(X > \ln c)$

1 $= P\left(\dfrac{X-m}{\sigma} > \dfrac{\ln c - m}{\sigma}\right) = P\left(Z > \dfrac{\ln c - m}{\sigma}\right)$, $Z \sim N(0,1)$

1 $= P\left(Z \le -\dfrac{\ln c - m}{\sigma}\right)$ since $P(Z > a) = P(Z < -a)$

1 $= P\left(Z \le \dfrac{m - \ln c}{\sigma}\right)$



Therefore, $P(e^X > c) = P\left(Z \le \frac{m - \ln c}{\sigma}\right)$

$$= \Phi\left(\frac{m - \ln c}{\sigma}\right). \quad 1$$

4 POINTS

(b) $$E\left[e^X \cdot I_{\{X>c\}}\right] = \int_{\{x:\, x>c\}} \frac{e^x \cdot e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}\, dx$$

Same as $\int_c^\infty$

1 Let $y = \frac{x-m}{\sigma} \Rightarrow dy = \frac{dx}{\sigma}$.

$$E\left[e^X \cdot I_{\{X>c\}}\right] = \int_{\{y:\, y > \frac{c-m}{\sigma}\}} \frac{e^{\sigma y + m} \cdot e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

The term $\sigma y + m - y^2/2 = -\frac{1}{2}(y-\sigma)^2 + m + \sigma^2/2$

1 $$E\left[e^X \cdot I_{\{X>c\}}\right] = e^{m+\sigma^2/2} \int_{\frac{c-m}{\sigma}}^{\infty} \frac{e^{-1/2(y-\sigma)^2}}{\sqrt{2\pi}}\, dy.$$

Perform another change of variable by letting $z = y - \sigma$, $dz = dy$.

1 $$E\left[e^X \cdot I_{\{X>c\}}\right] = e^{m+\sigma^2/2} \int_{\frac{c-m-\sigma^2}{\sigma}}^{\infty} \frac{e^{-1/2 z^2}}{\sqrt{2\pi}}\, dz$$

$$= e^{m+\sigma^2/2} \int_{-\infty}^{-\left(\frac{c-m-\sigma^2}{\sigma}\right)} \frac{e^{-1/2 z^2}}{\sqrt{2\pi}}\, dz,$$ using symmetry of the normal density

1 $$= e^{m+\sigma^2/2}\, \Phi\left(\frac{m+\sigma^2-c}{\sigma}\right).$$