SS 4521G/FM 9561B – Winter 2014

Solutions to Assignment No.2

Problems NOT required for submission

1. Consider the Black-Scholes-Merton price for the European option given by

$$
\begin{aligned}
c &= S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \\
\frac{\partial c}{\partial t} &= S\Phi'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\Phi'(d_2)\frac{\partial d_2}{\partial t}.
\end{aligned}
$$

From the claim (assigned exercise) given in the lecture, $S\Phi'(d_1) = Ke^{-r(T-t)}\Phi'(d_2)$.

Hence,

$$
\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}\Phi(d_2) + S\Phi'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right).
$$

Since

$$
\begin{aligned}
d_1 - d_2 &= \sigma\sqrt{T-t} \\
\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} &= \frac{\partial}{\partial t}(\sigma\sqrt{T-t}) \\
&= -\frac{\sigma}{2\sqrt{t-t}}.
\end{aligned}
$$

Consequently,

$$
\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}\Phi(d_2) - S\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}}.
$$

2. Suppose the strike price is 10.00. The option writer aims to be fully covered whenever the option is in the money and naked whenever it is out of the money. The option writer attempts to achieve this by buying the assets underlying the option as soon as the asset price reaches 10.00 from below and selling as soon as the asset price reaches 10.00

from above. The trouble with this scheme is that it assumes that when the asset price moves from 9.99 to 10.00, the next move will be a price above 10.00. In practice the next move might be back to 9.99. Similarly, it assumes that when the asset price moves from 10.01 to 10.00, the next move will be to a price below 10.00. In practice the next move might be back to 10.01. The scheme can be implemented by buying at 10.01 and selling at 9.99. However, it is not a good hedge. The cost of the trading strategy is zero if the asset price never reaches 10.00 and can be quite high if it reaches 10.00 many times. A good hedge has the property that its cost is always very close to the value of the option.

3. By examining the formula of the theta parameter, a theta of -0.1 means that if δt years pass with no change in either the stock price or its volatility, the value of the option declines by $0.1\delta t$.

4. The strategy costs the trader \$0.20 each time the stock is bought and sold. The total expected cost of the strategy, in present value terms, must be \$4. This means that the expected number of times the stock will be bought or sold is approximately 40. The expected number of times it will be bought is approximately 20 and the expected number of times it will be sold is also approximately 20. The buy and sell transactions can take place at any time during the life of the option. The above numbers are therefore only approximately correct because of the effects of discounting. Also, they assume a risk-neutral world.

5. To hedge an option position it is necessary to create the opposite option position synthetically. For example, to hedge a long position in a put it is necessary to create a short position in a put synthetically. It follows that the procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position.

6. Portfolio insurance involves creating a put option synthetically. It assumes that as soon as the portfolio's value declines by a small amount the portfolio manager's position is rebalanced by either (a) selling part of the portfolio, or (b) selling index futures. On 19 October 1987, the market declined so quickly that the sort of rebalancing anticipated in portfolio insurance schemes could not be accomplished.

7. The fund is worth \$300,000 times the value of the index. When the value of the portfolio falls by 5% (to \$342 million), the value of the S&P 500 also falls by 5% to 1140. The fund manager therefore requires European put options on 300,000 times the S&P 500 with exercise price 1140.

   (i) S_t=1200, K=1140, r=0.06, σ=0.30, $T-t$=0.50 and q=0.03. Hence,

$$\begin{aligned} d_1 &= \frac{\ln(1200/1140) + (0.06 - 0.03 + 0.3^2/2)(0.5)}{0.3\sqrt{0.5}} = 0.4186 \\ d_2 &= d_1 - 0.3\sqrt{0.5} = 0.2064 \end{aligned}$$

   $\Phi(d_1) = 0.6622$; $\Phi(d_2) = 0.5818$
   $\Phi(-d_1) = 0.3378$; $\Phi(-d_2) = 0.4182$

   The value of one put is
   $1140e^{-r(T-t)}\Phi(-d_2) - 1200e^{-q(T-t)}\Phi(-d_1)$
   $= 1140e^{(-0.06)(0.5)}(0.4182) - 1200e^{(-0.03)(0.5}(0.3378) = 63.40.$

   The total cost of the insurance is therefore 300,000(63.40)=\$19,020,000.

   (ii) From put-call parity
   $S_te^{-q(T-t)} + p_t = c_t + Ke^{-r(T-t)}$ or $p_t = c_t - S_te^{-q(T-t)} + Ke^{-r(T-t)}$. This shows that a put option can be created by selling (or shorting) $e^{-q(T-t)}$ of the index, buying a call option and investing the remainder at the risk-free rate of interest. Applying this situation under consideration, the fund manager should:
   1) Sell $360e^{-0.03(0.5)}$ = \$354.64 million of stock
   2) Buy call options on 300,000 times the S&P 500 with exercise price

of 1140 and maturity in six months.
3) Invest the remaining cash at the risk-free interest rate of 6% per annum.

This strategy gives the same result as buying put options directly.

(iii) The delta of one put option
$e^{-q(T-t)}[\Phi(d_1) - 1] = e^{-0.03(0.5)}(0.6622 - 1) = -0.3327.$

This indicates that 33.27% of the portfolio (i.e., \$119.77 million) should be initially sold and invested in risk-free securities.

8. The delta indicates that when the value of the euro exchange rate increases by \$0.01, the value of the bank's position increases by (0.01)(30,000)=\$300. The gamma indicates that when the euro exchange rate increases by \$0.01 the delta of the portfolio decreases by \$0.01(80,000)=800. For delta neutrality 30,000 euros should be shorted. When the exchange rate moves up to 0.93, we expect the delta of the portfolio to decrease by (0.93-0.90)(80,000)=2,400 so that it becomes 27,600. To maintain delta neutrality, it is therefore necessary for the bank to unwind its short position of 2,400 euros so that a net 27,600 have been shorted.

   As shown in the Hull's textbook (8th ed), Figure 18.8, page 390, when a portfolio is delta-neutral and has a negative gamma, a loss is experienced when there is a large movement in the underlying asset price. We can conclude that the bank is likely to have lost money.

9. For a call option on a non-dividend-paying stock

$$\begin{aligned}\Delta &= \Phi(d_1)\\ \Gamma &= \frac{\Phi'(d_1)}{S\sigma\sqrt{T-t}}\\ \Theta &= -\frac{S\Phi'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2).\end{aligned}$$

Hence, the left-hand side of the "Greeks" relation is:

$$
\begin{aligned}
&-\frac{S\Phi'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2) + rS\Phi(d_1) + \frac{1}{2}\sigma S\frac{\Phi'(d_1)}{\sqrt{T-t}} \\
&= r[S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)] \\
&= r\Pi.
\end{aligned}
$$

(b) For a put option on a non-dividend-paying stock

$$
\begin{aligned}
\Delta &= \Phi(d_1) - 1 = -\Phi(-d_1) \\
\Gamma &= \frac{\Phi'(d_1)}{S\sigma\sqrt{T-t}} \\
\Theta &= -\frac{S\Phi'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}\Phi(-d_2).
\end{aligned}
$$

Hence, the left-hand side of the "Greeks" relation is:

$$
\begin{aligned}
&-\frac{S\Phi'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}\Phi(-d_2) - rS\Phi(-d_1) + \frac{1}{2}\sigma S\frac{\Phi'(d_1)}{\sqrt{T-t}} \\
&= r[Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1)] \\
&= r\Pi.
\end{aligned}
$$

(c) For a portfolio of options, Π, Δ, Θ and Γ are the sums of their values for the individual options in the portfolio. It follows that the relation for the "Greeks" is true for any portfolio of European put and call options.

10. The heavier tail should lead to high prices, and therefore high implied volatilities for out-of-the-money (low-strike-price) puts. Similarly, the less heavy right tail should lead to low prices, and therefore low volatilities for out-of-the-money (high-strike-price) calls. A volatility smile where volatility is a decreasing function of strike price results.

11. With the notation used in the lecture,

$$\begin{aligned} c_t^{BS} + Ke^{-r(T-t)} &= p_t^{BS} + Se^{-q(T-t)} \\ c_t^{mkt} + Ke^{-r(T-t)} &= p_t^{mkt} + Se^{-q(T-t)}. \end{aligned}$$

It follows that $c_t^{BS} - c_t^{mkt} = p_t^{BS} - p_t^{mkt}$. In this case, $c_t^{mkt} = 3.00$; $c_t^{BS} = 3.50$ and $p_t^{BS} = 1.00$. It follows that p_t^{mkt} should be 0.50.

12. The crashophobia argument is an attempt to explain the pronounced volatility skew in equity market since 1987. This was the year equity markets shocked everyone by crashing more than 20% in one day. The argument is that traders are concerned about another crash and as a result increase the price of out-of-the-money puts. This creates the volatility skew.

13. The probability distribution of the stock price in one month is not lognormal. Possibly it consists of two lognormal distributions superimposed upon each other and is bimodal. Black-Scholes is clearly inappropriate because it assumes that the stock price at any future time is lognormal.

14. When the asset price is positively correlated with volatility, the volatility tends to increase as the asset price increases, producing less heavy left tails and heavier right tails. Implied volatility then increases with the strike price.

15. Jumps tend to make both tails of the stock price distribution heavier than those of the lognormal distribution. This creates a volatility smile and this is likely to be more pronounced for the three-month option.

16. In this case the probability distribution of the exchange rate has a thin left tail and a thin right tail relative to the lognormal distribution. Both out-of-the-money and in-the-money calls and puts can be expected to have lower implied volatilities than at-the-money calls and puts. This will give rise to a volatility frown.

17. There are a number of problems in testing an option pricing model empirically. These include the problem of obtaining synchronous data on stock prices and option prices, the problem of estimating the dividends that will be paid on the stock during the option's life, the problem of distinguishing between situations where the market is inefficient and situations where the option pricing model is incorrect, and the problems of estimating stock price volatility.

18. A deep-out-of-the-money option has a low value. Decreases in its volatility reduce its value. However, this reduction is small because the value can never go below zero. Increases in its volatility, on the other hand, can lead to significant percentage increases in the value of the option. The option does, therefore, have some of the same attributes as an option on volatility.

19. When plain vanilla call and put options are being priced, traders do use Black-Scholes model as an interpolation tool. They calculate implied volatilities for the options whose prices they can observe in the market. By interpolating between strike prices and between times to maturity, they estimate implied volatilities for other options. These implied volatilities are then substituted into Black-Scholes to calculate prices for these options. In practice much of the work in producing a table of values that generate the volatility surface in the over-the-counter (OTC) market is done by brokers. Brokers often act as intermediaries between participants in the OTC market and usually have more information on the trades taking place than any individual financial institution. The brokers provide a table of values generating a volatility

surface as a service to their clients.

20. ⋆ Assigned problem. Solution in a separate file.

21. ⋆ Assigned problem. Solution in a separate file.

22. ⋆ Assigned problem. Solution in a separate file.

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