FM 9521B

SOLUTIONS TO SS 4521G ASSIGNMENT NO. 2
Winter 2014

4 PTS

Required Problem 1 [Question 20]:

From the Black-Scholes-Merton formula

$$d_1 = \frac{\ln S_t/X + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and}$$

$$d_2 = \frac{\ln S_t/X + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Also, $d_2 = d_1 - \sigma\sqrt{T-t}$ so that

$$d_1^2 = d_2^2 + 2d_2\sigma\sqrt{T-t} + \sigma^2\sqrt{T-t}.$$

Thus,

$$S_t e^{-d_1^2/2} - Xe^{-r(T-t)-d_2^2/2}$$

1

$$= S_t e^{-d_2^2/2-\sigma\sqrt{T-t}\,d_2-\sigma^2/2(T-t)} - Xe^{-r(T-t)-d_2^2/2}$$

$$= e^{-d_2^2/2}\left[S_t e^{-d_2\sigma\sqrt{T-t}-\sigma^2/2(T-t)} - Xe^{-r(T-t)}\right]$$

1

$$= e^{-d_2^2/2}\left[S_t e^{-\ln S_t/X - r(T-t) + \sigma^2/2(T-t) - \sigma^2/2(T-t)} - Xe^{-r(T-t)}\right]$$

$$= e^{-d_2^2/2}\left[S_t \cdot e^{\ln X/S_t} e^{-r(T-t)} - Xe^{-r(T-t)}\right]$$

1

$$= e^{-d_2^2/2}\left[\cancel{S_t}\cdot\frac{X}{\cancel{S_t}} e^{-r(T-t)} - Xe^{-r(T-t)}\right]$$

$$= 0.$$

To show that $S_t \Phi'(d_1) = X e^{-r(T-t)} \Phi'(d_2)$,
we can equivalently show

$$S_t \Phi'(d_1) - X e^{-r(T-t)} \Phi'(d_2) = 0. \qquad (*)$$

(1)

LHS of $(*)$

$$= S_t \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} - X e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2}$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\underbrace{S_t e^{-d_1^2/2} - X e^{-r(T-t) - d_2^2/2}}_{=0 \text{ from the above}} \Big]$$

$$= 0.$$

Required Problem 2 [Question 21]:

The delta of the portfolio is

$$-1,000\times 0.50-500\times 0.80-2,000\times(-0.40)-500\times 0.70=-450$$ 0.5

The gamma of the portfolio is

$$-1,000\times 2.2-500\times 0.6-2,000\times 1.3-500\times 1.8=-6,000$$ 0.5

The vega of the portfolio is

$$-1,000\times 1.8-500\times 0.2-2,000\times 0.7-500\times 1.4=-4,000$$ 0.5

Let w_1 be the position in the first traded option (Option A) and w_2 be the position in the second traded option (Option B). We require

$$6,000=1.5w_1+0.5w_2$$

$$4,000=0.8w_1+0.6w_2$$ 2

The solution to this system of equations can easily be seen to be $w_1=3,200$, 1 $w_2=2,400$. The whole portfolio then has a delta of
1

$$-450+3,200\times 0.6+2,400\times 0.1=1,710$$ 1

Therefore the portfolio can be made delta, gamma and vega neutral by taking a long position in 3,200 of the first traded option, a long position in 2,400 of the second traded option and a short position of 1,710 in sterling.

1.5

Required Problem 3 [Question 22]:

[Note to student: This problem requires that important assumptions, steps in calculations and **formulae** employed must be adequately specified. It is not necessary to include the codes. however a detailed explanation of how the results are obtained will receive high marks.]

The results of the calculations are shown in the following table. For example, when the strike price is 34, the price of a call option with a volatility of 10% is 5.926, and the price of a call option when the volatility is 30% is 6.312. When there is a 60% chance of the first volatility and 40% of the sdecond, the price is $0.6 \times 5.926 + 0.4 \times 6.312 = 6.080$. The implied volatility given by this price is 23.21. The table shows that the uncertainty about volatility leads to a classic volatility smile similar to that in Figure 19.1 of the text. In general when volatility is stochastic with the stock price and volatility uncorrelated we get a pattern of implied volatilities similar to that observed for currency options.

1

Each implied volatility in the last column is found by employing either a trial-and-error approach or the Newton-Raphson method, which is an iterative procedure

1

based on the recursion $\sigma_{n+1} = \sigma_n - \dfrac{f(\sigma_n)}{f'(\sigma_n)}$, where σ_0 is assumed given, and $f(\sigma_n)$ is given by $f(\sigma_n) = g(\sigma_n)$ – observed market futures option price = 0. Here, $g(\sigma_n)$ is **the Black Scholes price for a call option on a futures contract**.

The $g(\sigma_n)$ is given by $g = e^{-rT}\,[F_0\,\Phi(d_1) - X\,\Phi(d_2)]$ where

1

$$d_1 = \frac{\ln\frac{F_0}{X} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

and $d_2 = d_1 - \sigma\sqrt{T}$.

Also, $f'(\sigma_n) = g'(\sigma_n)$ where $g'(\sigma) = F_0\sqrt{T}\,\Phi'(d_1)e^{-rT}$

[Note: For those using binomial instead of Black-Scholes formula, the risk-neutral probability q of an up movement needs to be correctly specified.] In this case, q = (1 – d)/(u-d), where u and d are the respective appreciation and depreciation factors.

[Note: Solutions are expected to include an illustration of how the first implied volatility (or any implied volatility) is obtained.]

Strike Price	*Call Price* *10% Volatility*	*Call Price* *30% Volatility*	*Weighted Price*	*Implied Volatility (%)*
34	5.926	6.312	6.080	23.21
36	3.962	4.749	4.277	21.03
38	2.128	3.423	2.646	18.88
40	0.788	2.362	1.418	18.00
42	0.177	1.560	0.730	18.80
44	0.023	0.988	0.409	20.61
46	0.002	0.601	0.242	22.43

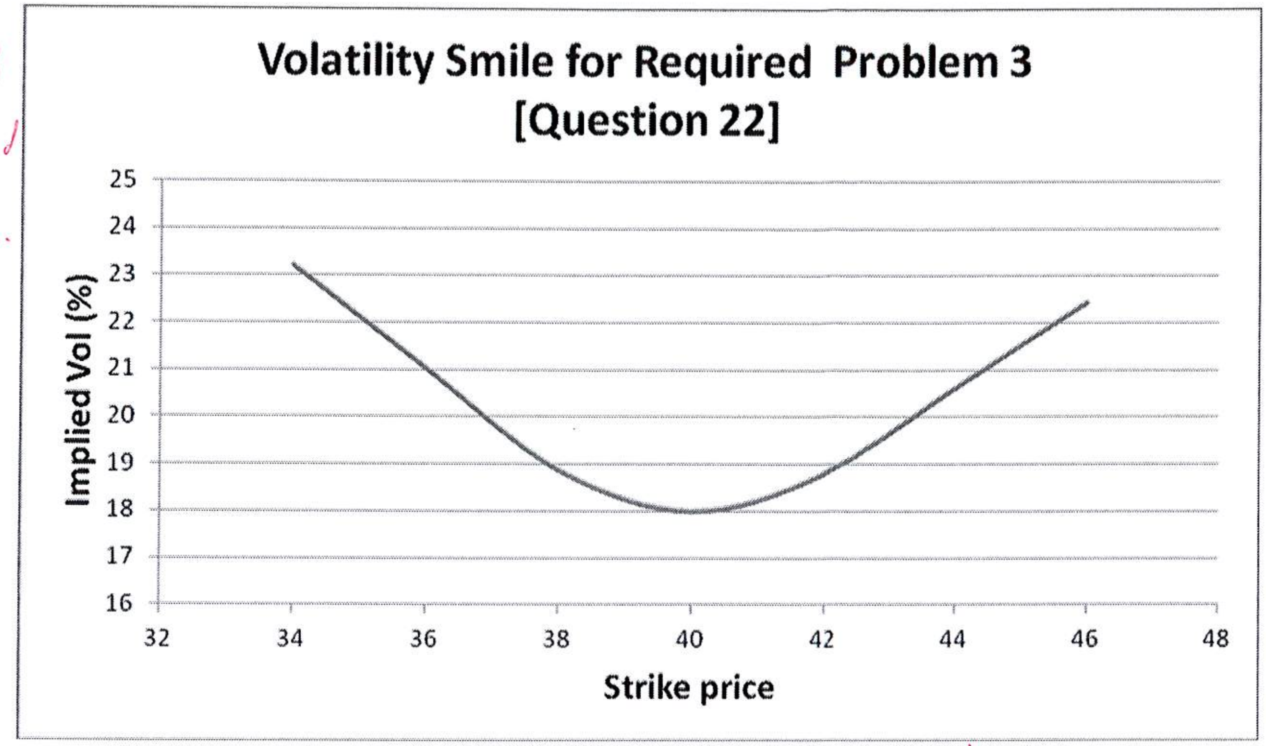
the same as the graph 2

1 1 1

Plotting the first and last columns of the table above produces the volatility smile as shown below.

2

for implied vol column in the table.



-0.1 for wrong/missing labels.