Statistical Sciences 4521G/FM 9521B

Solutions to Assignment No.3

Problems NOT required for submission

ESTIMATING VOLATILTIES

1. (Problem 22.2) The EWMA model produces a forecast of the daily variance rate for day n which is a weighted average of (i) the forecast for day $n-1$ and (ii) the square of the proportional change on day $n-1$. The GARCH (1,1) model produces a forecast of the daily variance for day n which is a weighted average of (i) the forecast for day $n-1$, (ii) the square of the proportional change on $n-1$ and (iii) a long run average variance rate. Whereas the EWMA has no mean reversion, GARCH (1,1) is consistent with a mean-reverting variance rate model.

2. (Problem 22.3) In this case $\sigma_{n-1} = 0.015$ and $u_{n-1} = 0.5/30 = 0.01667$. Thus, $\sigma_n^2 = 0.94(0.015)^2 + 0.06(0.01667)^2 = 0.0002281$. The volaitlity estimate on day n is therefore $\sqrt{0.0002281} = 0.015103$ or 1.5103%.

3. (Problem 22.6) The weight given to the long-run average variance rate is $1-\alpha-\beta$ and the long-rum average rate is $\omega/(1-\alpha-\beta)$. Increasing ω increases the long-run variance rate. Increasing α increases the weight given to the most recent data item, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate. Increasing β increases the weight given to the previous variance estimate, reduces the weight given to the long-run average variance rate and increases the level of the long-run average variance rate.

4. (Problem 22.7) The proportional daily change is -0.005/1.5000=-0.003333. The current daily variance estimate is $0.006^2 = 0.000036$. The new daily variance estimate is $0.9(0.000036)+0.1(0.003333)^2 = 0.000033511$. The new volatility is the square root of this. It is 0.00579 or 0.579%.

5. (Problem 22.8) With the usual notation $u_{n-1} = 20/1040 = 0.01923$ so that $\sigma_n^2 = 0.000002 + 0.06(0.01923)^2 + 0.92(0.01)^2 = 0.0001162$. Thus, $\sigma_n = 0.01078$. The new volatility estimate is therefore 1.078% per day.

6. (Problem 22.11) In this case, $\sigma_{x,n-1} = 0.01$ and $\sigma_{y,n-1} = 0.012$ and the most recent estimate of the covariance between the asset returns is $\text{cov}_{n-1} = (0.01)(0.012)(0.50) = 0.00006$. The variable $x_{n-1} = 1/30 = 0.03333$ and the variable $y_n = 1/50 = 0.02$. The new estimate of the covariance, cov_n, is $0.000001 + 0.04(0.03333)(0.02) + 0.94(0.00006) = 0.0000841$. The new estimate of the variance of the first asset, $\sigma_{x,n}^2$ is $0.000003 + 0.04(0.03333)^2 + 0.94(0.01)^2 = 0.0001414$ so that $\sigma_{x,n} = \sqrt{0.0001414} = 0.01189$ or 1.189%. The new estimate of the variance of the second asset, $\sigma_{y,n}^2$ is $0.000003 + 0.04(0.02)^2 + 0.94(0.012)^2 = 0.0001544$ so that $\sigma_{y,n} = \sqrt{0.0001544} = 0.01241$ or 1.242%. The new estimate of the correlation between the assets is therefore 0.0000841/[(0.01189)(0.01242)]=0.5

AMERICAN OPTIONS

7. (Problem 14.15) We have $D_1 = D_2 = 1$, $X(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1(0.1667)}) = 1.07$, and $X(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1(0.25)}) = 1.60$. Since $D_2 < X(1 - e^{-r(T-t_2)})$ and $D_1 < X(1 - e^{-r(t_2-t_1)})$, it is never optimal to exercise the call option early.

8. (Problem 14.21) Here, we have $D_1 = D_2 = 1.50$, $t_1 = 0.3333$, $t_2 = 0.8333$, $T = 1.25$, $r = 0.08$ and $X = 55$. So, $X(1 - e^{-r(T-t_2)}) = 55(1 - e^{-0.08(0.4167)}) = 1.80$. Hence, $D_2 < X(1 - e^{-r(T-t_2)})$. Also, $X(1 - e^{-r(t_2-t_1)}) = 55(1 - e^{-0.08(0.50)}) = 2.16$. Hence, $D_1 < X(1 - e^{-r(t_2-t_1)})$.

   It follows that the option should never be exercised early.

The present value of the dividends is $1.5e^{-0.3333(0.08)} + 1.5e^{-0.8333(0.08)} = 2.864$.

The option can be valued using the Black-Scholes pricing formula with $S_0 = 50 - 2.864 = 47.136, \quad X = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$. This gives a price of \$4.17.

EXOTIC DERIVATIVES

9. (Problem 25.1) A forward start option is an option that is paid for now but will start at some time in the future. The strike price is usually equal to the price of the asset at the time the option starts. A chooser option is an option where, at some time in the future, the holder chooses whether the option is a call or put.

10. (Problem 25.2) A lookback call provides a pay-off of $S_T - S_{\text{min}}$. A lookback put provides a payoff of $S_{\text{max}} - S_T$. A combination of a lookback call and lookback put therefore provides a payoff of $S_{\text{max}} - S_{\text{min}}$.

11. (Problem 25.7) The option is in the money when the asset price is less than the strike price. However, in these circumstances the barrier has been hit and the option has ceased to exist.

INTEREST-RATE DERIVATIVES

12. (Problem 13.9) The drift is $a(b - r_t)$. Thus, when the interest rate is above b the drift rate is negative, and when the interest rate is below b, the drift is positive. The interest rate is therefore continually pulled towards the level b. The rate at which it is pulled toward this level is a. A volatility is equal to c is superimposed upon the "pull" or the drift.

    Suppose $a = 0.4$, $b = 0.1$ and $c = 0.15$ and the current interest rate is 20% per annum. The interest rate is pulled towards the level of 10%

per annum. This can be regarded as a long-run average. The current drift is -4% per annum so that the expected rate at the end of one year is about 16% per annum. Superimposed upon the drift is a volatility of 15% per annum.

13. (Problem 13.11) The process followed by B, the bond price, is from Itō's lemma:

$$dB_t = \left[\frac{\partial B}{\partial x} a(x_0 - x_t) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x} s^2 {x_t}^2\right] dt + \frac{\partial B}{\partial x} sx dW_t.$$

Since x is the yield to maturity, $B = e^{-x(T-t)}$, the required partial derivatives are

$$\begin{aligned} \frac{\partial B}{\partial x} &= xe^{-x(T-t)} = xB \\ \frac{\partial B}{\partial x} &= -(T-t)e^{-x(T-t)} = -(T-t)B \\ \frac{\partial^2 B}{\partial x^2} &= (T-t)^2 e^{-x(T-t)} = (T-t)^2 B. \end{aligned}$$

Hence,

$$dB_t = \left[-a(x_0 - x)(T-t) + x + \frac{1}{2} s^2 {x_t}^2\right] B_t dt - sx_t(T-t)B_t dW_t.$$

PROBLEMS REQUIRED FOR SUBMISSION

14. ($\star$) (Problems 22.17–22.18) Assigned problem. Solution in a separate file.

15. ($\star$) Assigned problem. Solution in a separate file.

16. ($\star$) (Problem 25.32) Assigned problem. Solution in a separate file.

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