SS4521G - 27–31 January 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed in the context of stock price processes evolving on a two-period binomial tree model in discrete time:

1. Cameron-Girsanov-Martin Theorem (given in class without proof): Suppose W_t is a $P-$Brownian motion and γ_t is $\mathcal{F}_t-$adapted process satisfying the boundedness (or growth) condition $E^P\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] < \infty$. Then there exists a measure Q such that
   (i) Q is equivalent to P;
   (ii) $\frac{dQ}{dP} := \Lambda_T = \exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right)$; and
   (iii) $\widetilde{W}_t = W_t + \int_0^t \gamma_s ds$ is a $Q-$Brownian motion. Or, $W_t = \widetilde{W}_t - \int_0^t \gamma_s ds$ is a $Q-$Brownian motion with drift $-\gamma_t$ at time t.

   The CGM theorem is a powerful tool for controlling the drift of any process.

2. We discussed the solution to this problem: Suppose X_t is a stochastic process with dynamics $dX_t = \mu_t dt + \sigma_t dW_t$. Is there a measure Q such that the drift of the process X under Q is $v_t dt$ instead of $\mu_t dt$? Consequently, we gave justification by invoking CMG why under the risk-neutral measure Q the drift rate of the geometric Brownian motion is r instead of μ.

3. To find the corresponding European put price p_t, we simply use the put-call parity given by the relation $p_t + S_t = c_t + Xe^{-r(T-t)}$, where

p_t, c_t, and S_t are the respective put, call and stock prices at time t; and X and $T-t$ are the respective strike price and amount of time to option's contract maturity. Put-call parity and its proof were covered in SS 3520.

4. When dividends are involved (assuming that the stock pays a dividend yield at the rate q), the modified put-call parity is $p_t^q + S_t e^{-q(T-t)} = c_t^q + Ke^{-r(T-t)}$, where p_t^q and c_t^q are the respective put and call prices on a dividend-paying stock. This is due to the fact that the payment of a dividend yield causes the stock price to drop by an amount of the dividend. The price $c^q(t, S_t)$ of a European call option on a stock that pays dividends at rate q is given by

$$c^q(t, S_t) = S_t e^{-q(T-t)} \Phi(d_1^q) - Ke^{-r(T-t)} \Phi(d_2^q)$$

where $d_1^q = \dfrac{\ln \frac{S_t}{K} + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$.

and $d_2^q = \dfrac{\ln \frac{S_t}{K} + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1^q - \sigma\sqrt{T-t}$.

Note that when the stock pays a dividend yield at rate q, we modify the put-call parity and the Black-Scholes formula by replacing S_t by $S_t e^{-q(T-t)}$.

5. Options on currencies (e.g., AUD, GBP, CAD, JPY, CHF & EUR against USD) are actively traded in both over-the-counter and exchange markets. Recall from SS 3520 that a foreign currency is analogous to a stock paying a known dividend yield. The owner of a foreign currency receives a "dividend yield" equal to the risk-free rate r_f, in the foreign currency. Assuming that the stochastic process for a foreign currency is the same as that for a stock paying a dividend equal to the foreign risk-free rate, the corresponding European option price formula and the put-call parity are obtained by replacing q with r_f.

6. In the risk-neutral world (i.e., under measure Q), individuals/market agents are risk-neutral. There is no compensation for risk and the expected return on all securities is the risk-free rate.

7. We discussed certain properties of the Black-Scholes formula:
   (i) When the stock price S_t becomes large, a call option is almost certain to be exercised and this becomes very similar to a forward contract with a delivery price K. It is expected that the call price will be $S_t - Ke^{-r(T-t)}$. We showed that this fact is consistent with the Black-Scholes formula.

   (ii) When the stock price S_t becomes very large, the price of a European put option will approach to zero. Again, we showed that this is consistent with the behaviour of the Black-Scholes formula.

   (iii) The case when the volatility σ approaches zero was considered. This means the stock is virtually riskless and it grows at the risk-free rate r to $S_t e^{r(T-t)}$ at time $T > t$. The pay-off of the option is then $\max(S_t e^{r(T-t)} - K, 0)$, where K is the strike price, and the option value would be $\max(S_t - Ke^{-r(T-t)}, 0)$. It was demonstrated that this fact is consistent with the Black-Scholes-Merton formula.

8. An important parameter in the pricing and hedging of options is delta (Δ). It is the number of units of the stock we should hold for each option shorted in order to create a riskless hedge. It turns out that this is the ratio of the change in the price of a stock option to the change in the price of the underlying stock.

9. The construction of a riskless hedge is referred to as *delta hedging.* The delta of a call option is positive whereas the delta of a put is negative.

Hedging schemes

10. Option traders attempt to make a portfolio immune to small changes in the price of the underlying asset in the next small time interval. The so-called delta hedging examines the value delta (Δ) of a derivative security. This value is given by $\Delta = \dfrac{\partial f}{\partial S}$ where f is the value of the portfolio and S is the price of the underlying asset. Traders also look at gamma and vega defined by the following:

    Γ (gamma):=rate of change of the value of the portfolio with respect to delta. That is,
    $\Gamma = \dfrac{\partial}{\partial S}\left(\dfrac{\partial f}{\partial S}\right) = \dfrac{\partial \Delta}{\partial S} = \dfrac{\partial^2 f}{\partial S^2}$.

    ϑ (vega):=rate of change of the value of the portfolio with respect to asset's volatility, so that
    $\Lambda = \dfrac{\partial f}{\partial \sigma}$.

    Other sensitivity parameters that practitioners consider are:

    Θ (theta): = the rate of change of the portfolio with respect to the passage of time. In particular,
    $\Theta = \dfrac{\partial f}{\partial t}$ and

    ρ (rho):= the rate of change with respect to the risk-free rate r, so that
    $\rho = \dfrac{\partial f}{\partial r}$.

    Exercise: With the notation used in the Black-Scholes formula, show that $Se^{-\frac{d_1^2}{2}} - Ke^{-r(T-t)-\frac{d_2^2}{2}} = 0$.

11. Using the result of the above exercise and if c is the price of a European call option, we showed that the delta for the call option is given by $\Delta = \dfrac{\partial c}{\partial S} = \Phi(d_1)$. For the put option, $\dfrac{\partial p}{\partial S} = \Phi(d_1) - 1$.

12. An interesting result relating hedging/sensitivity parameters follows from the Black-Sholes-Merton PDE and is given by

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rc,$$

where c is the European option price.

13. **Monte-Carlo (MC) simulation technique applied to the pricing of a European call option.**
Given a stochastic differential equation $dS_t = \mu S_t dt + \sigma S_t dW_t$, its discretised version is $S_{k+1} - S_k = \mu S_k \Delta t_k + \sigma S_k \epsilon \sqrt{\Delta t_k}$, where $\Delta t_k = t_{k+1} - t_k$, $\epsilon \sim N(0,1)$ and the sense of the equality is understood to in be in law or distribution. Given an initial value S_0, this recursion of the discretised version of the SDE (or its solution) can be employed to generate sample paths of the stochastic process.