SS4521G – 09 January 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

07 January 2014 lecture - Cancelled due to snowstorm (extreme wind-chill).

The following concepts/theories were covered/reviewed in the context of stock price processes evolving on a two-period binomial tree model in discrete time:

1. Probability space $(\Omega, \mathcal{F}, P)$.

2. Random variable $X : \Omega \to I\!R$.

3. Stochastic process $X_t(\omega)$ or $X(t, \omega)$, which is a family of random variables (RVs), where $\omega \in \Omega$.

4. The concept of probability measure was also discussed. Intuitively, this refers to a set of probabilities.

5. The filtration process. A filtration is a sequence of non-decreasing sub-$\sigma-$fields or sub-$\sigma-$algebras. A $\sigma-$field $\mathcal{F}$ is a non-empty collection of subsets of a sample space Ω such that $\Omega \in \mathcal{F}$ and it is closed under set complementation and countable union.

6. All models that will be considered in this course are assumed to be defined on a probability space equipped with a filtration.

7. A contingent claim on the binomial tree is a function of the nodes at a claim-time horizon T. Note that this is also a function of the filtration $\{\mathcal{F}_T\}$. Claims can either be (sample) path-dependent or path-independent.

8. Conditional expectation operator $E^P[\ \cdot\ |\mathcal{F}_k]$: This extends the idea of expectation to two parameters, namely a measure P and a history up to time k. This is an expectation along the latter portion of paths which have initial segment $\{\mathcal{F}_k\}$; if it makes it easier you may view the node attained at time k as the new root of the binomial tree and take expectations of future claims from there.

9. A stochastic process S is a martingale with respect to a measure Q and a filtration $\{\mathcal{F}_k\}$ if (i) $E^Q[S_k] < \infty$ and (ii) $E^Q[S_k|\mathcal{F}_j] = S_j, \quad \forall\ j \leq k$. It simply means that the future expected value at time k of the process S under Q conditional on its history until time j is merely the process value at time j.

10. We may occasionally need to use the fact that for $i \leq j$ and claim X,

$$E^P\left[E^P[X|\mathcal{F}_j]|\mathcal{F}_i\right] = E^P[X|\mathcal{F}_i].$$

    In other words, conditioning on the history up to time j and then conditioning on the history up to earlier time i is the same as just conditioning originally up to time i. This result is called the *tower law.*

    We showed in class that for any claim X, the conditional expectation process $E^P[X|\mathcal{F}_k]$ is always a $P-$martingale.

11. We considered the random walk process under some measure, say, P. In particular, for $n \in \mathbb{Z}^+$, the random walk is a binomial process $W_n(t)$ satisfying the following conditions: (i) $W_n(0) = 0$, (ii) layer spacing of $\frac{1}{n}$, (iii) up and down jumps equal and of equal size $\frac{1}{\sqrt{n}}$ and (iv) under measure P, the up and down probabilities everywhere are equal to $\frac{1}{2}$.

    If $X_1, X_2, \ldots$ is a sequence of independent binomial RVs taking values +1 and -1 with equal prob, then value of W_n at the ith step has the representation

$$W_n\left(\frac{i}{n}\right) = W_n\left(\frac{i-1}{n}\right) + \frac{X_i}{\sqrt{n}}, \quad \text{for } i \geq 1.$$

12. We showed in class that $W_n(1)$ is normal with mean 0 and variance 1. In general, by using the central limit theorem, we showed that $W_n(t) \sim N(0, t)$ as $n \to \infty$.

13. Random walk has the property that its future movements away from a particular position are independent of where that position is and indeed, independent of its entire history of movements up to that time. Moreover, the increment/displacement $W_n(s+t) - W)n(s)$ has mean 0 and variance t. Consequently, W_n converges towards a Brownian motion. These observations on $W_n(s+t) - W_n(s)$ form the basis of the definition for the Brownian motion below.

14. The process $W = \{W_t : t \geq 0\}$ is a $P-$Brownian motion (BM) if (i) W_t is continuous and $W_0 = 0$, (ii) $W_t \sim N(0, t)$ under P and (iii) $W_{t+s} - W_s \sim N(0, t)$ under P and independent of $\mathcal{F}_s$, the history of

what the process attains up to time s.

15. BM is also called Wiener process. It is a one-dimensional Gaussian process.

16. Although a BM W is continuous everywhere, it is nowhere differentiable. BM will eventually hit any value no matter how large or negative. It may be a million units above the horizontal axis but it will (with probability one) be back again to zero at some later time. Once a BM hits a value, it hits again infinitely often, and then again from time to time in the future.

17. We started looking at how BM can be used as a model for stock price evolution. We also looked at the dynamics of a stochastic process driven by a Brownian motion by considering the so-called stochastic differential equation. Further discussions and illustrative examples will be provided next meeting.