SS4521G - 03–07 March 2014

SUMMARY OF IMPORTANT POINTS DISCUSSED IN THE LECTURE

The following concepts/theories were covered/reviewed:

1. We discussed how risk-neutral distribution can be backed out using option prices. Here, we employ the Leibniz rule of differentiation. An illustration on how to recover the risk-neutral distribution using market call option prices was demonstrated in the lecture. This idea was first proposed by Breeden and Litzenberger in a paper published in Journal of Business (1978).

   Volatility/correlation estimation

2. The estimation of volatilities and correlations is very important in quantitative finance. Volatilities/correlations of market variables are used in the calculation of value-at-risk and volatility, in particular, is an important input in option pricing.

   We wish to study how historical data can be used to produce estimates of the current levels of volatilities and correlations. The models we shall study include (i) Exponentially weighted moving average (EWMA), (ii) Autoregressive conditional heteroscedasticity (ARCH) and (iii) Generalised autoregressive conditional heteroscedasticity (GARCH). These models are designed to keep track of the variations in the volatility or correlation through time and capture volatility persistence (i.e., periods of low volatility and periods of high volatility).

3. **Notation:**
   σ_n :=volatility of a market variable on day n.
   σ_n^2 :=variance rate
   S_i :=value of market variable at the end of day i.

$u_i := \ln\left(\frac{S_i}{S_{i-1}}\right)$ =continuously compounded return during day i and between $i-1$ and i.

For value-at-risk calculation, note that we would use $u_i = \frac{S_i - S_{i-1}}{S_{i-1}}$ =% change in the market variable.

4. When estimating the current level of σ_n, it is more reasonable to give more weights to the more recent data. Let α_i be the weights assigned to the observation i days ago. So, we must have $\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-1}^2$. The constraints are $\alpha_i < \alpha_j$ for $i > j$ and $\sum_{i=1}^{m} \alpha_i = 1$.

5. ARCH (m). This model was proposed by Engle (1982) who was one of the two winners of the Nobel Prize in Economic Sciences in 2003. The model's formulation is as follows:

   Suppose the variance rate possesses a long-run average. This could take the form $\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$ where V_L =long-run variance rate and γ =weight given to V_L. Here, we must have $\gamma + \sum_{i=1}^{m} \alpha_i = 1$. Note that σ_n^2 is based on (i) long-run average, i.e., we have a mean-reverting volatility process and (ii) m observations.

   The conventional form $\sigma_n^2 = \omega + \sum_{i=1}^{m} \alpha_i u_{n_i}^2$ is used when parameters need to be estimated where $\omega = \gamma V_L$.

6. For the EWMA, the weight α_i decreases exponentially as we move back through time. That is, $\alpha_{i+1} = \lambda \alpha_i$, where $0 < \lambda < 1$. The EWMA

model can then be written as

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2.$$

It was shown in class that the above equation (for the EWMA model) can be re-expressed further as

$$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-i}^2 + \lambda^m\sigma_{n-m}^2.$$

The term $\lambda^m\sigma_{n-m}^2$ is sufficiently small for large m (and can be ignored). Thus, this model is the same as the one described in #4 with $\alpha_i = (1-\lambda)\lambda^{i-1}$. Clearly, the rate declines as we move back through the past.

7. The GARCH(p, q) model can be employed to compute the variance rate σ_n^2 using the p most recent observations $u_i's$ and q most recent estimates of the variance rates. This model was proposed by Bollerslev (1986). In particular, $\sigma_n^2 = \omega + \sum_{i=1}^{p}\alpha_i u_{n-i}^2 + \sum_{i=1}^{q}\beta_i\sigma_{n-i}^2$ where $\omega = \gamma V_L$. Here, α_i is the weight given to u_{n-i}^2, β_i is the weight given to σ_{n-i}^2, γ is the weight given to V_L, which is the targeted long-run average variance rate.

8. To estimate the GARCH model parameters, we re-visited the method of maximum likelihood estimation (MLE). In particular, we considered the estimation of the variance v of a random variable X given m observations. We assume that $X \sim N(0, v)$ and u_i is the realisation of X. We formed the likelihood function $L(\cdot)$ for X and for a given parameter θ (v in this case), we have

$$L(\theta) = \prod_{i=1}^{m}\left[\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{u_i^2}{2v}\right)\right].$$

The MLE value is the best estimate of v that maximises the above equation for $L(\theta)$.

We note that maximising an expression is equivalent to maximising the log of the expression; this is the reason why we maximised $\ln L(\theta)$. The MLE for v is given by $\widehat{v} = \frac{\sum_{i=1}^{m} u_i^2}{m}$.

9. To estimate the parameters α, β and γ in the GARCH model, one can use the MLE technique. The estimation of parameters via MLE is to tantamount to solving the following optimisation problem:

   maximise $\sum_{i=1}^{n} \left(-\ln v_i - \frac{u_i^2}{v_i} \right)$
   subject to
   $v_i := \sigma_i^2 = \omega + \alpha u_{i-1}^2 + \beta \sigma_{i-1}^2$
   $\gamma + \alpha + \beta = 1.$

   In the above, $\omega = \gamma V_L$, where V_L is usually set as the target rate as in #7.